Math 2270-004 Week 4 notes
We will not necessarily finish the material from a given day's notes on that day. We may also add or subtract some material as the week progresses, but these notes represent an in-depth outline of what we plan to cover. These notes cover material in 1.8-1.9, 2.1-2.2.

Mon Jan 29
- 1.8-1.9 Matrix and linear transformations

Announcements: continuing from below, I could compute $T(\vec{x}(t))$ directly:

$$= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} t \\ 1+2t \end{bmatrix} = \begin{bmatrix} -t \\ 1+2t \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} -1 \\ 2 \end{bmatrix} \checkmark$$

$$\left[\begin{array}{l} T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v}) \\ T(c\vec{u}) = cT(\vec{u}) \end{array}\right.$$

Warm-up Exercise: Consider $T: \mathbb{R}^2 \to \mathbb{R}^2$

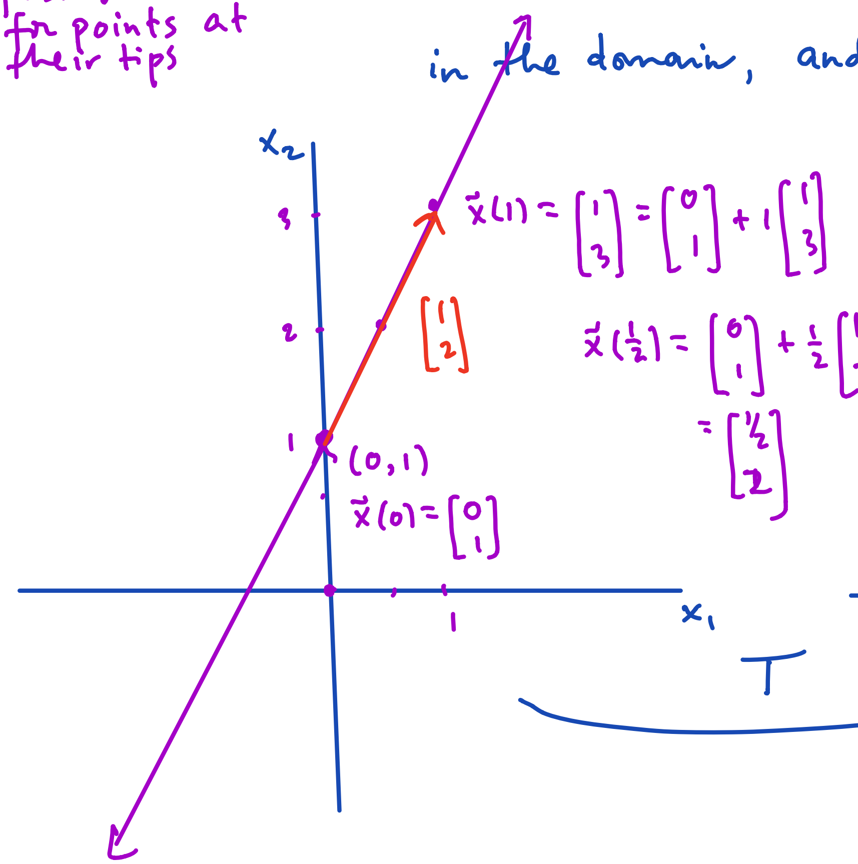
'til 1:00

$$T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

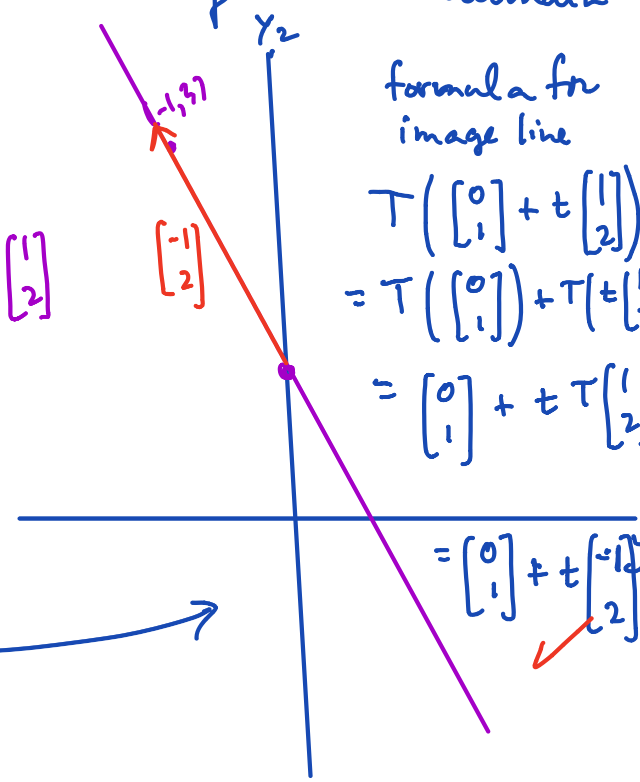
(reflection across the x_2-axis)

remember, these are position vectors for points at their tips

- sketch the parametric line $\vec{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $t \in \mathbb{R}$ in the domain, and its image in the codomain



T



formula for image line

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) + T\left(t\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + t\,T\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} -1 \\ 2 \end{bmatrix} \checkmark$$

Recall from Friday that

<u>Definition</u>: A function T which has domain equal to $\mathbb{R}^n$ and whose range lies in $\mathbb{R}^m$ is called a *linear transformation* if it transforms sums to sums, and scalar multiples to scalar multiples. Precisely, $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is *linear* if and only if

$$\begin{aligned} T(\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}}) &= T(\underline{\boldsymbol{u}}) + T(\underline{\boldsymbol{v}}) \qquad && \forall\, \underline{\boldsymbol{u}}, \underline{\boldsymbol{v}} \in \mathbb{R}^n \\ T(c\,\underline{\boldsymbol{u}}) &= c\,T(\underline{\boldsymbol{u}}) && \forall\, c \in \mathbb{R}, \underline{\boldsymbol{u}} \in \mathbb{R}^n \ . \end{aligned}$$

<u>Notation</u> In this case we call $\mathbb{R}^m$ the *codomain*. We call $T(\underline{\boldsymbol{u}})$ the *image of* $\underline{\boldsymbol{u}}$. The *range* of T is the collection of all images $T(\underline{\boldsymbol{u}})$, for $\underline{\boldsymbol{u}} \in \mathbb{R}^n$.

<u>Important connection to matrices:</u> Each matrix $A_{m \times n}$ gives rise to a linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$, namely

$$T(\underline{\boldsymbol{x}}) := A\,\underline{\boldsymbol{x}} \qquad \forall\, \underline{\boldsymbol{x}} \in \mathbb{R}^n.$$

This is because, as we checked last week, matrix transformation satisfies the linearity axioms:

$$\begin{aligned} A(\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}}) &= A\,\underline{\boldsymbol{u}} + A\,\underline{\boldsymbol{v}} \qquad && \forall\, \underline{\boldsymbol{u}}, \underline{\boldsymbol{v}} \in \mathbb{R}^n \\ A(c\,\underline{\boldsymbol{u}}) &= c\,A\,\underline{\boldsymbol{u}} && \forall\, c \in \mathbb{R}, \underline{\boldsymbol{u}} \in \mathbb{R}^n \end{aligned}$$

Recall, we verified this by using the column notation $A = \left[\underline{\boldsymbol{a}}_1, \underline{\boldsymbol{a}}_2, \ldots\, \underline{\boldsymbol{a}}_n\right]$, writing

$$\underline{\boldsymbol{u}} = \begin{bmatrix} u_1 \\ u_2 \\ : \\ u_n \end{bmatrix}, \underline{\boldsymbol{v}} = \begin{bmatrix} v_1 \\ v_2 \\ : \\ v_n \end{bmatrix},$$

and computing

$$\begin{aligned} A(\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}}) &= \left[\underline{\boldsymbol{a}}_1, \underline{\boldsymbol{a}}_2, \ldots\, \underline{\boldsymbol{a}}_n\right](\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}}) \\ &= \left(u_1 + v_1\right)\underline{\boldsymbol{a}}_1 + \left(u_2 + v_2\right)\underline{\boldsymbol{a}}_2 + \ldots + \left(u_n + v_n\right)\underline{\boldsymbol{a}}_n \\ &= \left(u_1\underline{\boldsymbol{a}}_1 + u_2\underline{\boldsymbol{a}}_2 + \ldots + u_n\underline{\boldsymbol{a}}_n\right) + \left(v_1\underline{\boldsymbol{a}}_1 + v_2\underline{\boldsymbol{a}}_2 + \ldots + v_n\underline{\boldsymbol{a}}_n\right) \\ &= A\,\underline{\boldsymbol{u}} + A\,\underline{\boldsymbol{v}}\,. \end{aligned}$$

And,

$$\begin{aligned} A(c\,\underline{\boldsymbol{u}}) &= \left[\underline{\boldsymbol{a}}_1, \underline{\boldsymbol{a}}_2, \ldots\, \underline{\boldsymbol{a}}_n\right](c\,\underline{\boldsymbol{u}}). \\ &= cu_1\underline{\boldsymbol{a}}_1 + cu_2\underline{\boldsymbol{a}}_2 + \ldots + cu_n\underline{\boldsymbol{a}}_n = c\,A\,\underline{\boldsymbol{u}} \\ &= c\,A\,\underline{\boldsymbol{u}}\ . \end{aligned}$$

Geometry of linear transformations: (We saw this illustrated in a concrete example on Friday)

Theorem: For each matrix $A_{m \times n}$ and corresponding linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, given by

$$T(\underline{\boldsymbol{x}}) := A\,\underline{\boldsymbol{x}} \qquad \forall\, \underline{\boldsymbol{x}} \in \mathbb{R}^n$$

- (parallel) lines in the domain $\mathbb{R}^n$ are transformed by T into (parallel) lines or points in the codomain $\mathbb{R}^m$.

- 2-dimensional planes in the domain $\mathbb{R}^n$ are transformed by T into (parallel) planes, (parallel) lines, or points in the codomain $\mathbb{R}^m$.

proof:

- A parametric line through a point with position vector $\underline{\boldsymbol{p}}$ and direction vector $\underline{\boldsymbol{u}}$ in the domain can be expressed as the set of point having position vectors

$$\underline{\boldsymbol{p}} + t\,\underline{\boldsymbol{u}}, \quad t \in \mathbb{R}.$$

The image of this line is the set of points

$$A(\underline{\boldsymbol{p}} + t\,\underline{\boldsymbol{u}}) = A\,\underline{\boldsymbol{p}} + t\,A\,\underline{\boldsymbol{u}}, \quad t \in \mathbb{R}$$

"$A(\vec{p}) + A(t\vec{u})$"

which is either a line through $A\,\underline{\boldsymbol{p}}$ with direction vector $A\,\underline{\boldsymbol{u}}$, when $A\,\underline{u} \neq \underline{\boldsymbol{0}}$, or the point $A\,\underline{\boldsymbol{p}}$ when $A\,\underline{\boldsymbol{u}} = \underline{\boldsymbol{0}}$. Since parallel lines have parallel direction vectors, their images also will have parallel direction vectors, and therefore be parallel lines.

parallel line: $\vec{q} + t\vec{u}$

$A(\vec{q} + t\vec{u}) = A(\vec{q}) + t\,A(\vec{u})$

is parallel to 1st line if $A(\vec{u}) \neq \vec{0}$.

- A parametric (2-dimensional) plane through a point with position vector $\underline{\boldsymbol{p}}$ and independent direction vectors $\underline{\boldsymbol{u}}$, $\underline{\boldsymbol{v}}$ in the domain can be expressed as the set of point having position vectors

$$\underline{\boldsymbol{p}} + t\,\underline{\boldsymbol{u}} + s\,\underline{\boldsymbol{v}}, \quad t, s \in \mathbb{R}.$$

The image of this line is the set of points

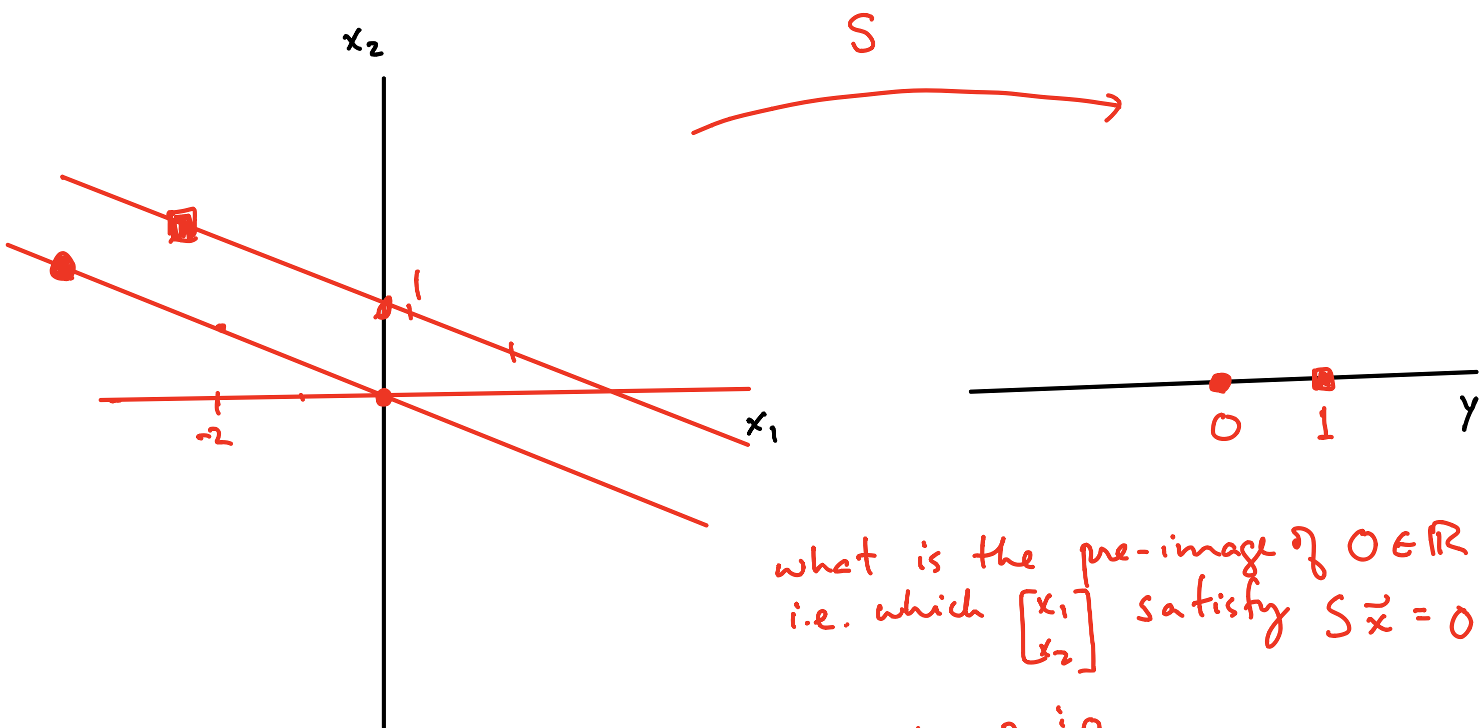
$$A(\underline{\boldsymbol{p}} + t\,\boldsymbol{u} + s\,\underline{\boldsymbol{v}}) = A\,\underline{\boldsymbol{p}} + t\,A\,\underline{\boldsymbol{u}} + s\,A\,\underline{\boldsymbol{v}}, \quad t, s \in \mathbb{R}$$

which is either a plane through $A\,\underline{\boldsymbol{p}}$ with independent direction vectors $A\,\underline{\boldsymbol{u}}$, $A\,\underline{\boldsymbol{v}}$, or a line through point $A\,\underline{\boldsymbol{p}}$ if $A\,\underline{\boldsymbol{u}}$, $A\,\underline{\boldsymbol{v}}$ are dependent but not both zero, or the point $A\,\underline{\boldsymbol{p}}$ if $A\,\underline{u} = A\,\underline{v} = \underline{\boldsymbol{0}}$. Since parallel planes can be expressed with the same direction vectors, their images also will have parallel direction vectors, and therefore be parallel.

Exercise 1) Consider the linear transformation $S : \mathbb{R}^2 \to \mathbb{R}$ given by

$$S\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) := \begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [x_1 + 2x_2].$$

Make a geometric sketch that indicates what the transformation does. In this case the interesting behavior is in the domain.



what is the pre-image of $0 \in \mathbb{R}$

i.e. which $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ satisfy $S\vec{x} = 0$

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \end{array}\right]$$

$x_1 = -2t$
$x_2 = t$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t\begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

preimage of 1

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \end{array}\right]$$

$x_1 = 1 - 2t$
$x_2 = t$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

For the rest of today we'll consider linear transformations from $\mathbb{R}^2 \to \mathbb{R}^2$. (See the transformed goldfish in text section 1.9 - it's worth it.) We write the standard basis vectors for $\mathbb{R}^2$ as

$$\underline{\boldsymbol{e}}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ \underline{\boldsymbol{e}}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$= x_1 \vec{e}_1 + x_2 \vec{e}_2$$

If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is linear, then

$$\begin{aligned} T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) &= T(x_1 \underline{\boldsymbol{e}}_1 + x_2 \underline{\boldsymbol{e}}_2) \\ &= T(x_1 \underline{\boldsymbol{e}}_1) + T(x_2 \underline{\boldsymbol{e}}_2) \\ &= x_1 T(\underline{\boldsymbol{e}}_1) + x_2 T(\underline{\boldsymbol{e}}_2) \end{aligned}$$

$$= [T(\underline{\boldsymbol{e}}_1) \ \ T(\underline{\boldsymbol{e}}_1)] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}!$$

In other words, if $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation, it's actually a matrix transformation. And the columns of the matrix, in order, are $T(\underline{\boldsymbol{e}}_1), T(\underline{\boldsymbol{e}}_2)$. (The same idea shows that every linear $T : \mathbb{R}^n \to \mathbb{R}^m$ is actually a matrix transformation....and that the columns of the matrix are T applied to the standard basis vectors in the domain $\mathbb{R}^n$.)

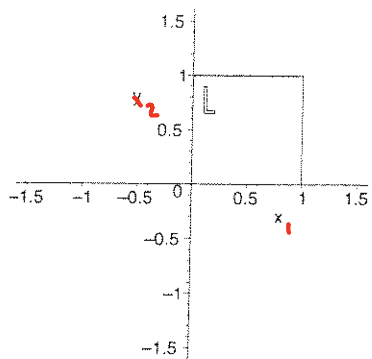
In the following diagrams the unit square in the domain, together with an "L" to keep track of whether the transformation involves a reflection or not, is on the left. On the right is the image of the square and the L under the transformation. Find the matrix for T!!! (Also note, that because parallel lines transform to parallel lines, grids go to transformed grids, so once you know how the unit square transforms, you know everything about the transformation. Or, to put it another way, once you know $T(\underline{\boldsymbol{e}}_1), T(\underline{\boldsymbol{e}}_2)$, you know the matrix for T so you know how every position vector transforms.)

§2.2 Important linear transformations $T:\mathbb{R}^2 \to \mathbb{R}^2$

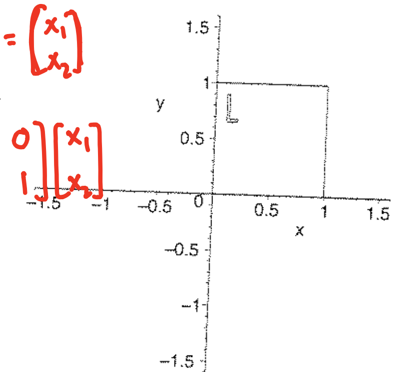
Bretscher likes to use an "L" to represent the geometry of the transformation.
I like to use "L boxes" instead ~ will tie into Maple project I on fractals.

Find the matrix formulas for these transformations!

If T is linear, it's a matrix transformation with matrix $\left[T(\vec{e}_1)\ T(\vec{e}_2)\right]$



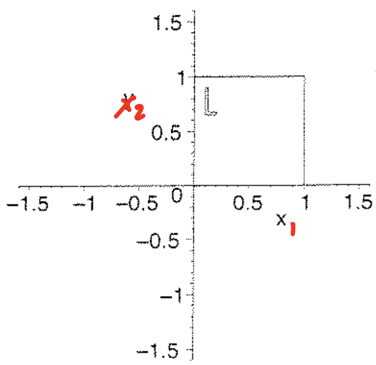
$$T_1 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$



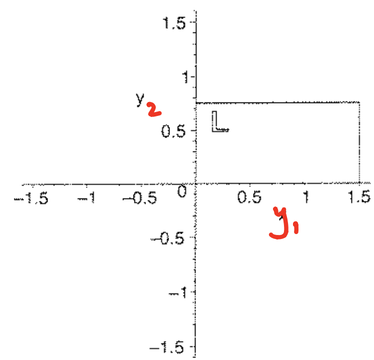
identity transformation

$T(\vec{e}_1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $T(\vec{e}_2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ✓



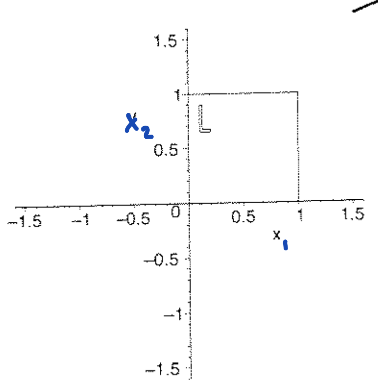
T_2

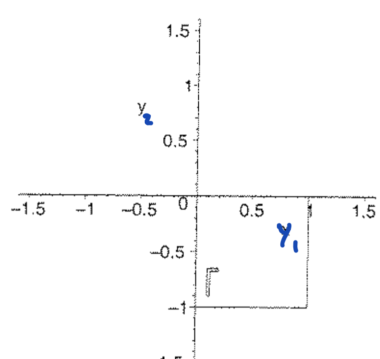


stretch by 1.5 horizontally & .75 vertically (from origin)

$$T_2 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \left[T_2(\vec{e}_1) \ \vdots \ T_2(\vec{e}_2) \right] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1.5 & 0 \\ 0 & .75 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1.5 x_1 \\ .75 x_2 \end{bmatrix}$$

T_3





reflect across x-axis

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$