Fri Jan 26

- 1.8 introduction to linear transformations

matrix transformations.

important in CS visualization

Announcements:

Warm-up Exercise: Definitions! precise & super consise

'til 12:57

$\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n\}$ linearly independent

$c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n = \vec{0} \iff c_1 = c_2 = \ldots = c_n = 0$

(i.e. no $\vec{v}_j$ is a linear combination of the others)

linearly dependent

$c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n = \vec{0}$ for at least one choice of weights that are not all zero

(at least one $\vec{v}_j$ is a linear combo of the others)

span$\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n\}$

all possible linear combinations of $\{\vec{v}_1, \vec{v}_2 \ldots \vec{v}_n\}$

i.e. all possible $x_1\vec{v}_1 + x_2\vec{v}_2 + \ldots + x_n\vec{v}_n$ for $x_j \in \mathbb{R}$.

1.8 Introduction to linear transformations.

Definition: A function T which has domain equal to $\mathbb{R}^n$ and whose range lies in $\mathbb{R}^m$ is called a *linear transformation* if it transforms sums to sums, and scalar multiples to scalar multiples. Precisely, $\boxed{T : \mathbb{R}^n \rightarrow \mathbb{R}^m}$ is *linear* if and only if

$$
\begin{aligned}
T(\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}}) &= T(\underline{\boldsymbol{u}}) + T(\underline{\boldsymbol{v}}) && \forall\ \underline{\boldsymbol{u}}, \underline{\boldsymbol{v}} \in \mathbb{R}^n \\
T(c\,\underline{\boldsymbol{u}}) &= c\,T(\underline{\boldsymbol{u}}) && \forall\ c \in \mathbb{R}, \underline{\boldsymbol{u}} \in \mathbb{R}^n\ .
\end{aligned}
$$

name of function

domain

Notation In this case we call $\boxed{\mathbb{R}^m \text{ the } \textit{codomain}.}$ We call $T(\underline{\boldsymbol{u}})$ the *image of* $\underline{\boldsymbol{u}}$. The *range* of T is the collection of all images $T(\underline{\boldsymbol{u}})$, for $\underline{\boldsymbol{u}} \in \mathbb{R}^n$.

Important connection to matrices: Each matrix $A_{m \times n}$ gives rise to a linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$, namely

$$T(\underline{\boldsymbol{x}}) := A\,\underline{\boldsymbol{x}} \qquad \forall\ \underline{\boldsymbol{x}} \in \mathbb{R}^n.$$

This is because, as we checked last week,

$$
\begin{aligned}
A(\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}}) &= A\,\underline{\boldsymbol{u}} + A\,\underline{\boldsymbol{v}} && \forall\ \underline{\boldsymbol{u}}, \underline{\boldsymbol{v}} \in \mathbb{R}^n \\
A(c\,\underline{\boldsymbol{u}}) &= c\,A\,\underline{\boldsymbol{u}} && \forall\ c \in \mathbb{R}, \underline{\boldsymbol{u}} \in \mathbb{R}^n
\end{aligned}
$$

Exercise 1) Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be defined by

$A_{3 \times 2}$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) := \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. = x_1 \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 5 \\ 7 \end{bmatrix}$$

1a Find $T(\underline{\boldsymbol{u}})$ for $\underline{\boldsymbol{u}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$. $= 2 \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} - 1 \begin{bmatrix} -3 \\ 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -9 \end{bmatrix}$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) := \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

$$x_1\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} + x_2\begin{bmatrix} -3 \\ 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix} \text{ or } \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$$

1b Is there an $\underline{\boldsymbol{x}} \in \mathbb{R}^2$ whose image under T is $\underline{\boldsymbol{b}} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$? Is $\underline{\boldsymbol{x}}$ unique?

yes

$$\vec{x} = \begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix}$$

$$\left[\begin{array}{cc|cc} 1 & -3 & 3 & 3 \\ 3 & 5 & 2 & 2 \\ -1 & 7 & -5 & 5 \end{array}\right]$$

1c Is $\underline{\mathbf{c}} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$ in the range of T?

no

inconsistent

Hints: The following reduced row echelon computation may be helpful for b, c:

$$\left[\begin{array}{cc|cc} 1 & -3 & 3 & 3 \\ 3 & 5 & 2 & 2 \\ -1 & 7 & -5 & 5 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cc|cc} 1 & 0 & 3/2 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) := \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} + x_2\begin{bmatrix} -3 \\ 5 \\ 7 \end{bmatrix}$$

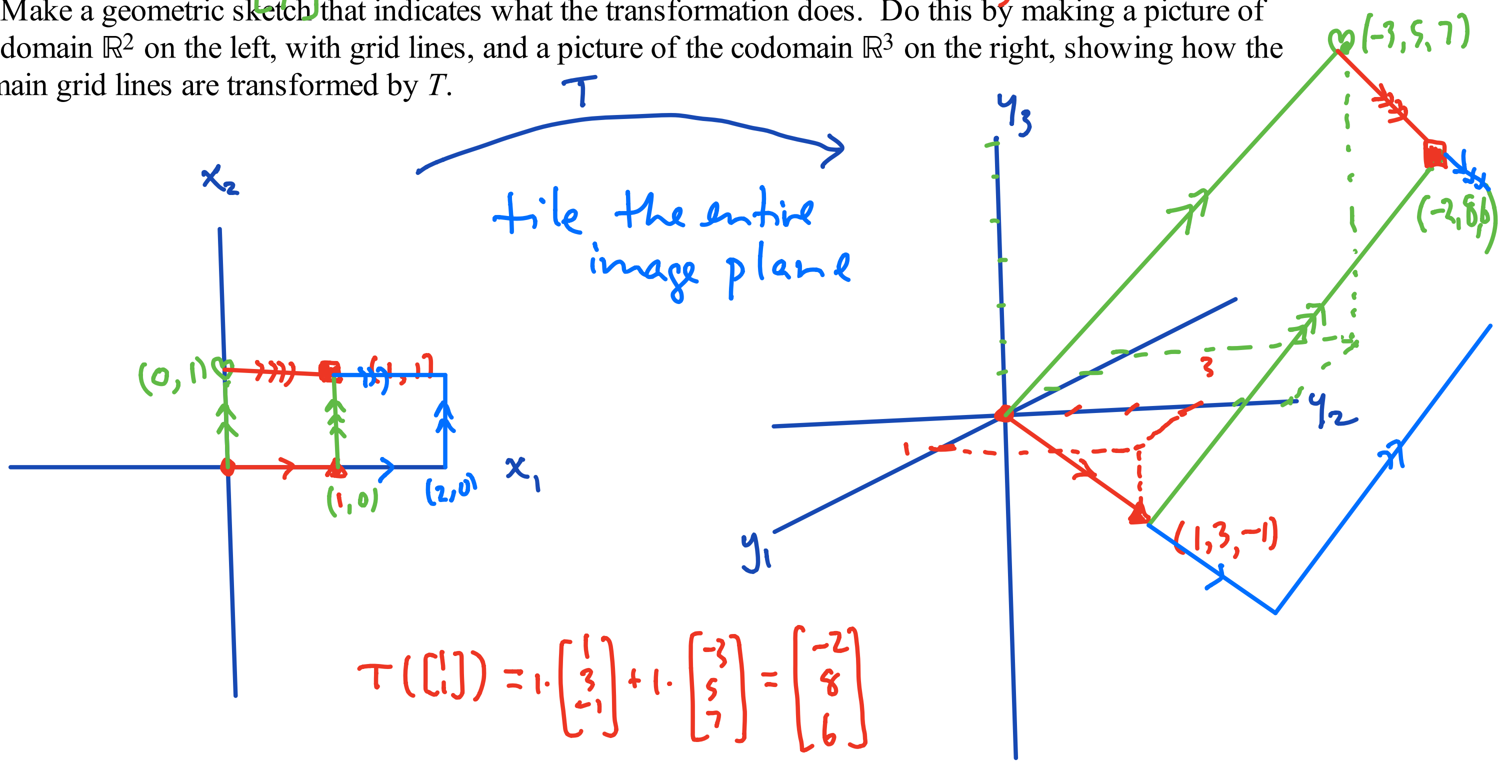
1d Describe the range of T as the span of two vectors. $= \text{span}\left\{\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ 5 \\ 7 \end{bmatrix}\right\}$

1d2) geometrically plane in $\mathbb{R}^3$

$T\left(x_2\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = x_2\begin{bmatrix} -3 \\ 5 \\ 7 \end{bmatrix}$

$T\left(x_1\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = x_1\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$ 1st col.

1e Make a geometric sketch that indicates what the transformation does. Do this by making a picture of the domain $\mathbb{R}^2$ on the left, with grid lines, and a picture of the codomain $\mathbb{R}^3$ on the right, showing how the domain grid lines are transformed by T.



$T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = 1\cdot\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} + 1\cdot\begin{bmatrix} -3 \\ 5 \\ 7 \end{bmatrix} = \begin{bmatrix} -2 \\ 8 \\ 6 \end{bmatrix}$

Exercise 2) Consider the linear transformation $S:\mathbb{R}^2 \rightarrow \mathbb{R}$ given by

$$S\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) := \begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Make a geometric sketch that indicates what the transformation does. In this case the interesting behavior is in the domain.