Note that the set of vectors $\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n\}$ is linearly dependent if and only if there are non-zero solutions \underline{c} to the homogeneous matrix equation

A <u>c</u> = <u>0</u>

for the matrix $A = \begin{bmatrix} \underline{v}_1 & \underline{v}_2 & \dots & \underline{v}_n \end{bmatrix}$ having the given vectors as columns. Thus all linear independence/dependence questions can be answered using reduced row echelon form.

12:58

WARM-UP 'til

Exercise 3) Show that the vectors

	1		-1		[-1]
$\underline{\mathbf{v}}_1 =$	0 2	$, \underline{v}_2 =$	2 0	$, \underline{v}_{3} =$	6 4

are linearly dependent (even though no two of them are scalar multiples of each other). What does this mean geometrically about the span of these three vectors? Find all dependencies



Hint: You might find this computation useful: Find all dependencies.

$$\begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 6 \\ 2 & 0 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0$$

Exercise 4) Are the vectors

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$$\mathbf{\underline{\nu}}_{1} = \begin{bmatrix} 1\\0\\2 \end{bmatrix}, \ \mathbf{\underline{\nu}}_{2} = \begin{bmatrix} -1\\2\\0 \end{bmatrix}, \ \mathbf{\underline{\nu}}_{3} = \begin{bmatrix} 1\\1\\1 \end{bmatrix}$$

linearly independent of dependent? Hint:

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \\ 2 & 0 & 1 \end{bmatrix} \stackrel{\bullet}{O} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \stackrel{\bullet}{O} \cdot \stackrel{\bullet}{\longrightarrow} \stackrel{\bullet}{\longrightarrow} \stackrel{\bullet}{O} \cdot \stackrel{\bullet}{\longrightarrow} \stackrel{\bullet}{\to} \stackrel{\bullet}{\to} \stackrel$$

dependency eqt is

$$c_{1} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + c_{2} \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + c_{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_{1} \\ c_{2} \\ c_{3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Exercise 5a) Why must more than three vectors in \mathbb{R}^3 be linearly dependent?

<u>5c</u>) If you are given a set of exactly *n* vectors in \mathbb{R}^n how can you check whether or not they are linearly independent? How does your criterion compare to the condition that will guarantee that the *n* vectors span \mathbb{R}^n ?

$$c_{1}\vec{v}_{1} + c_{2}\vec{v}_{2} + \dots + c_{n}\vec{v}_{n} = \vec{0}$$

$$h \begin{cases} \begin{bmatrix} \vec{v}_{1} & \vec{v}_{2} & \cdots & \vec{v}_{n} \end{bmatrix} \begin{bmatrix} c_{1} \\ c_{2} \\ \vdots \\ c_{n} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$h \end{cases}$$

$$for \{\vec{v}_{1}, \vec{v}_{2}, \dots, \vec{v}_{n}\} \text{ to be}$$

$$\lim_{n \to \infty} \begin{bmatrix} 1 \\ meanly \\$$

Wed Jan 24

• 1.7 Linear dependence/independence continued, and why each matrix has a unique reduced row echelon form

de pendeny ?'s are homogeneous matrix eqtu ?'s

Exercise 1) Consider the homogeneous matrix equation $A \underline{x} = \underline{0}$, with the matrix A (and its reduced row echelon form) shown below:

$$A := \begin{bmatrix} 1 & 2 & 0 & 1 & 1 \\ 2 & 4 & 1 & 4 & 1 \\ -2 & -4 & 0 & -2 & -2 \end{bmatrix} \bigcirc (0, -3) = \begin{bmatrix} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \bigcirc (0, -3) = \begin{bmatrix} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \bigcirc (0, -3) = \begin{bmatrix} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \bigcirc (0, -3) = \begin{bmatrix} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \bigcirc (0, -3) = \begin{bmatrix} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \bigcirc (0, -3) = \begin{bmatrix} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \bigcirc (0, -3) = \begin{bmatrix} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \bigcirc (0, -3) = \begin{bmatrix} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \bigcirc (0, -3) = \begin{bmatrix} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \bigcirc (0, -3) = \begin{bmatrix} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Find and express the solutions to this system in linear combination form. Note that you are finding all of the dependencies for the collection of vectors that are the columns of *A*, namely the set

$$\begin{bmatrix} \underline{\nu}_{1} = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}, \underline{\nu}_{1} = \begin{bmatrix} 2 \\ 4 \\ -4 \end{bmatrix}, \underline{\nu}_{3} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \underline{\nu}_{4} = \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}, \underline{\nu}_{5} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \end{bmatrix}.$$

$$c_{1} = -2p - r - t$$

$$c_{2} = free = p$$

$$c_{3} = -2r + t$$

$$c_{4} = r \quad free$$

$$c_{5} = t \quad free$$

$$c_$$

Exercise 2) Focus on the idea that solutions to homogeneous matrix equations correspond exactly to linear dependencies between the columns of the matrix. Now, since the solution set to a homogeneous linear system does not change as you do elementary row operations to the augmented matrix, column dependencies also do not change. Therefore the vectors that span the space of homogeneous solutions in Exercise 1 are encoding the key column dependencies in \mathbb{R}^3 , for both the original matrix, and for the reduced row echelon form.

Check this, by reading off "easy" column dependencies in the reduced matrix; seeing that they are also dependencies in the original matrix; and that they correspond to the basis of the homogeneous solution space. Magic! We will use this magic in important interesting ways, later in the course.

$$A := \begin{bmatrix} 1 & 2 & 0 & 1 & 1 \\ 2 & 4 & 1 & 4 & 1 \\ -2 & -4 & 0 & -2 & -2 \\ \mathbf{v} & \mathbf{v} & \mathbf{v} & \mathbf{v} & \mathbf{v} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Exercise 3) (This exercise explains why each matrix has only one reduced row echelon form, no matter what sequence of elementary row operations one uses to find it. We didn't have the tools to explain why this fact was true earlier in the chapter) Let $B_{4 \times 5}$ be a matrix whose columns satisfy the following dependencies:

• $col_1(B) \neq \underline{0}$ (i.e. is independent) $col_2(B) = 3 \ col_1(B)$ $col_3(B)$ is independent of column 1 $col_4(B)$ is independent of columns 1,3. $col_5(B) = -3 \ col_1(B) + 2 \ col_3(B) - col_4(B)$.

What is the reduced row echelon form of *B*?