5) homogeneous and nonhomogeneous systems

5a) The *homogeneous* matrix equation, $A\underline{\mathbf{x}} = \underline{\mathbf{0}}$ is always consistent.

5b) If $A\underline{\mathbf{x}} = \underline{\boldsymbol{b}}$ is also consistent, then the solution set to $A\underline{\mathbf{x}} = \underline{\boldsymbol{b}}$ is a translation of the solution set to $A\underline{\mathbf{x}} = \underline{\mathbf{0}}$. In other words, the solution set of $A\underline{\mathbf{x}} = \underline{\boldsymbol{b}}$ is the set of all vectors

$$\underline{\boldsymbol{w}} = \underline{\boldsymbol{p}} + \underline{\boldsymbol{v}}_h$$

where $\underline{\boldsymbol{p}}$ is a particular solution to $A\underline{\mathbf{x}} = \underline{\boldsymbol{b}}$ and $\underline{\boldsymbol{v}}_h$ is any solution of the homogeneous equation

$$A\underline{\mathbf{x}} = \underline{\boldsymbol{0}}.$$

Tuesday warmup, from Monday's notes

Exercise 1) I gave an abstract explanation for why 5b is true on Friday. We can see it more concretely if we understand how this exercise below generalizes: A double augmented matrix for $A\underline{\mathbf{x}} = \underline{\mathbf{0}}$ and $A\underline{\mathbf{x}} = \underline{\boldsymbol{b}}$ and its reduced row echelon form are shown. Find and express the homogeneous and non-homogeneous solutions in linear combination form.

'til 12:58

$$\left[\begin{array}{cccc|c:c} 2 & -1 & 1 & 5 & 0 & 1 \\ -3 & 2 & -1 & -8 & 0 & 0 \\ 1 & 0 & 1 & 2 & 0 & 2 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cccc|c:c} 1 & 0 & 1 & 2 & 0 & 2 \\ 0 & 1 & 1 & -1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

A $\vec{0} \,\vdots\, \vec{b}$ x_3 free x_4 free

$$A\vec{x}=\vec{0}: \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -x_3-2x_4 \\ -x_3+x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix} = s\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \ s,t \in \mathbb{R}$$

$$A\vec{x}=\vec{b}: \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2-x_3-2x_4 \\ 3-x_3+x_4 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \\ 0 \end{bmatrix} + \underbrace{x_3\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}}$$

$$= \vec{p} + \vec{v}_H$$

1.6 Some applications of matrix equations.

Exercise 2) Balance the following chemical reaction equation, for the burning of propane:

$$\overset{1}{\overset{\shortparallel}{x_1}}\, C_3H_8 + \overset{5}{\overset{\shortparallel}{x_2}}\, O_2 \rightarrow \overset{3}{\overset{\shortparallel}{x_3}}\, CO_2 + \overset{4}{\overset{\shortparallel}{x_4}}\, H_2O$$

$$\begin{array}{c} C \rightarrow \\ H \rightarrow \\ O \rightarrow \end{array} \quad x_1\begin{bmatrix}3\\8\\0\end{bmatrix} + x_2\begin{bmatrix}0\\0\\2\end{bmatrix} = x_3\begin{bmatrix}1\\0\\2\end{bmatrix} + x_4\begin{bmatrix}0\\2\\1\end{bmatrix}$$

$$x_1\begin{bmatrix}3\\8\\0\end{bmatrix} + x_2\begin{bmatrix}0\\0\\2\end{bmatrix} - x_3\begin{bmatrix}1\\0\\2\end{bmatrix} - x_4\begin{bmatrix}0\\2\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}3 & 0 & -1 & 0\\8 & 0 & 0 & -2\\0 & 2 & -2 & -1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}.$$

Hint: after you set up the problem, the following reduced row echelon form computation will be helpful:

$$\left[\begin{array}{cccc|c}3 & 0 & -1 & 0 & 0\\8 & 0 & 0 & -2 & 0\\0 & 2 & -2 & -1 & 0\end{array}\right] \xrightarrow{rref} \left[\begin{array}{cccc|c}1 & 0 & 0 & -\frac{1}{4} & 0\\0 & 1 & 0 & -\frac{5}{4} & 0\\0 & 0 & 1 & -\frac{3}{4} & 0\end{array}\right]$$

$x_1 = \frac{1}{4}x_4$

$x_2 = \frac{5}{4}x_4$

$x_3 = \frac{3}{4}x_4$

$x_4 =$ free

$$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = x_4\begin{bmatrix}\frac{1}{4}\\\frac{5}{4}\\\frac{3}{4}\\1\end{bmatrix} \qquad \text{Let } x_4 = 4: \begin{bmatrix}1\\5\\3\\4\end{bmatrix}$$

Exercise 3) Consider the following traffic flow problem (from our text): What are the possible flow patterns, based on the given information and that the streets are one-way, so none of the flow numbers can be negative?

EXAMPLE 2 The network in Figure 2 shows the traffic flow (in vehicles per hour) over several one-way streets in downtown Baltimore during a typical early afternoon. Determine the general flow pattern for the network.

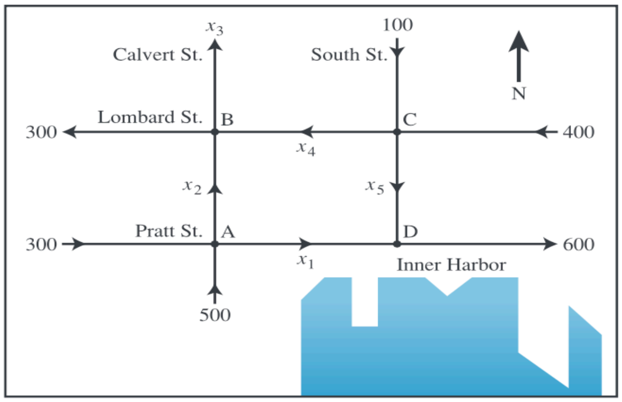

FIGURE 2 Baltimore streets.

@A: flow in = flow out

$800 = x_1 + x_2$

B: $x_2 + x_4 = 300 + x_3$

C: $500 = x_4 + x_5$

D: $x_1 + x_5 = 600.$

$$\begin{aligned} x_1 + x_2 \qquad\qquad &= 800 \\ x_2 - x_3 + x_4 \quad &= 300 \\ x_4 + x_5 &= 500 \\ x_1 \qquad\qquad + x_5 &= 600 \end{aligned}$$

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 800 \\ 0 & 1 & -1 & 1 & 0 & 300 \\ 0 & 0 & 0 & 1 & 1 & 500 \\ 1 & 0 & 0 & 0 & 1 & 600 \end{array}\right]$$

Hint: If you set up the flow equations for intersections A, B, C, D in that order, the following reduced row echelon form computation may be helpful:

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 800 \\ 0 & 1 & -1 & 1 & 0 & 300 \\ 0 & 0 & 0 & 1 & 1 & 500 \\ 1 & 0 & 0 & 0 & 1 & 600 \end{array}\right] \rightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 1 & 600 \\ 0 & 1 & 0 & 0 & -1 & 200 \\ 0 & 0 & 1 & 0 & 0 & 400 \\ 0 & 0 & 0 & 1 & 1 & 500 \end{array}\right]$$

$x_1 = 600 - x_5$

$x_2 = 200 + x_5$

$x_3 = 400$

$x_4 = 500 - x_5$

$x_5 =$ free

Tues Jan 23

- 1.7 linear dependence and independence. Connections to reduced row echelon form.

Announcements:

1.7 1, 5, 7 (15, 17, 27, 28, 31, 39, 42)

- finish Monday's notes : 1.6 applications
- start §1.7 on linear independence (continue tomorrow)

Warm-up Exercise:

1.7 When we are discussing the span of a collection of vectors $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$ we would like to know that we are being efficient in describing this span, and not wasting any free parameters because of redundancies in the vectors. For example, the most efficient way to describe a plane in $\mathbb{R}^3$ is as the span of exactly two vectors, rather than as the span of three or more. This has to do with the concept of "linear independence":

Definition:

not dependent

a) An indexed set of vectors $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n\}$ is said to be linearly independent if no one (1) of the vectors is a linear combination of (some) of the other vectors. The concise way to say this is that the only way $\underline{\mathbf{0}}$ can be expressed as a linear combination of these vectors,

$$c_1\underline{\boldsymbol{v}}_1 + c_2\underline{\boldsymbol{v}}_2 + \ldots + c_n\underline{\boldsymbol{v}}_n = \underline{\mathbf{0}},$$

is for all of the weights $c_1 = c_2 = \ldots = c_n = 0$.

start

b) An indexed set of vectors $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n\}$ is said to be linearly dependent if at least one of these (1) vectors *is* a linear combination of (some) of the other vectors. The concise way to say this is that there *is* some way to write $\underline{\mathbf{0}}$ as a linear combination of these vectors

(2)

$$c_1\underline{\boldsymbol{v}}_1 + c_2\underline{\boldsymbol{v}}_2 + \ldots + c_n\underline{\boldsymbol{v}}_n = \underline{\mathbf{0}}$$

where *not all* of the $c_j = 0$. (We call such an equation a linear dependency. Note that if we have any such linear dependency, then any $\underline{\boldsymbol{v}}_j$ with $c_j \neq 0$ is a linear combination of the remaining $\underline{\boldsymbol{v}}_k$ with $k \neq j$. We say that such a $\underline{\boldsymbol{v}}_j$ is linearly dependent on the remaining $\underline{\boldsymbol{v}}_k$.)

(1) means some $\vec{v}_j = d_1\vec{v}_1 + d_2\vec{v}_2 + \cdots + d_n\vec{v}_n$ (no $d_j\vec{v}_j$ term) not all d_j's are zero

$\Downarrow$

so $d_1\vec{v}_1 + d_2\vec{v}_2 + \cdots + d_n\vec{v}_n - \vec{v}_j = \vec{0}$ i.e. (2) holds.

(2) If $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = \vec{0}$ with not all weights $= 0$

assume $c_j \neq 0$ (pick j)

then $c_j\vec{v}_j = \underbrace{-c_1\vec{v}_1 - c_2\vec{v}_2 - \cdots - c_n\vec{v}_n}_{\text{no } \vec{v}_j \text{ term.}}$

$$\vec{v}_j = -\frac{c_1}{c_j}\vec{v}_1 - \frac{c_2}{c_j}\vec{v}_2 - \cdots - \frac{c_n}{c_j}\vec{v}_n$$

Note: The only set of a single vector $\{\underline{\boldsymbol{v}}_1\}$ that is dependent is if $\underline{\boldsymbol{v}}_1 = \underline{\mathbf{0}}$. The only sets of two non-zero vectors, $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2\}$ that are linearly dependent are when one of the vectors is a scalar multiple of the other one. For more than two vectors the situation is more complicated.

dependency eqtn $c_1\vec{v}_1 = \vec{0}$ is there a soltn $c_1 \neq 0$

$c_1 \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ if any $a_j \neq 0$ then $c_1 = 0$

so if $\vec{v} \neq \vec{0}$, $\{\vec{v}\}$ is indep.

Exercise 1a) Is this set linearly dependent or independent? $\left\{\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\}$?

$c_1 \begin{bmatrix} 0 \\ 0 \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ for $c_1 \neq 0$ YES

so $\left\{\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\}$ is dependent

1b) Is this set of vectors linearly dependent or independent? $\left\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ -3 \end{bmatrix}\right\}$?

$\vec{v}_1 \quad \vec{v}_2$

$\vec{v}_2 = -3\vec{v}_1$

(or $3\vec{v}_1 + \vec{v}_2 = \vec{0}$)

Example

The set of vectors $\left\{ \underline{\boldsymbol{v}}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \underline{\boldsymbol{v}}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \underline{\boldsymbol{v}}_3 = \begin{bmatrix} -2 \\ 8 \end{bmatrix} \right\}$ in $\mathbb{R}^2$ is linearly dependent because, as we showed when we were introducing vector equations (and as we can quickly recheck),

$$-3.5\begin{bmatrix} 1 \\ -1 \end{bmatrix} + 1.5\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 8 \end{bmatrix}.$$

① $\vec{v}_3$ is linearly dependent on $\vec{v}_1$ & $\vec{v}_2$

We can also write this linear dependency as

$$-3.5\underline{\boldsymbol{v}}_1 + 1.5\underline{\boldsymbol{v}}_2 - \underline{\boldsymbol{v}}_3 = \underline{\boldsymbol{0}}$$

② 2^{nd} way to say $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ are dependent

(or any non-zero multiple of that equation.)

Exercise 2) For a linearly independent set of vectors $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n\}$, every vector $\underline{\boldsymbol{v}}$ in their span can be written as $\underline{\boldsymbol{v}} = d_1\underline{\boldsymbol{v}}_1 + d_2\underline{\boldsymbol{v}}_2 + \ldots + d_n\underline{\boldsymbol{v}}_n$ uniquely, i.e. for exactly one choice of weights $d_1, d_2, \ldots d_n$. This is not true if vectors are dependent. Explain these facts. (You can illustrate these facts with the vectors in the example above.)

Note that the set of vectors $\{\underline{v}_1, \underline{v}_2, \ldots \underline{v}_n\}$ is linearly dependent if and only if there are non-zero solutions $\underline{c}$ to the homogeneous matrix equation

$$A\,\underline{c} = \mathbf{0}$$

for the matrix $A = [\underline{v}_1\ \underline{v}_2\ \ldots\ \underline{v}_n]$ having the given vectors as columns. Thus all linear independence/dependence questions can be answered using reduced row echelon form.

Exercise 3) Show that the vectors

$$\underline{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix},\ \underline{v}_2 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix},\ \underline{v}_3 = \begin{bmatrix} -1 \\ 6 \\ 4 \end{bmatrix}$$

are linearly dependent (even though no two of them are scalar multiples of each other). What does this mean geometrically about the span of these three vectors?

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Hint: You might find this computation useful:

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 2 & 6 & 0 \\ 2 & 0 & 4 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$