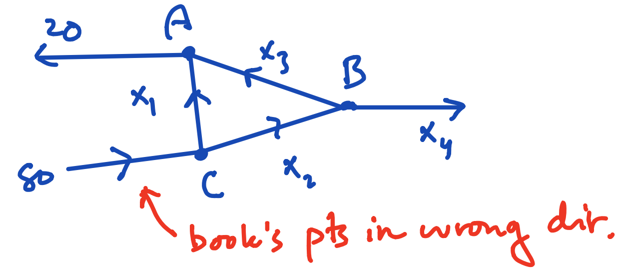
Math 2270-004 Week 3 notes
We will not necessarily finish the material from a given day's notes on that day. We may also add or subtract some material as the week progresses, but these notes represent an outline of what we plan to cover. These notes cover material in 1.5-1.8.

Mon Jan 22

- 1.5-1.6 review of facts we know, and some applications of systems of linear equations.

Announcements: wrong arrows on 1.6.11 :



Warm-up Exercise: Vocabulary "quiz"! Define 'til 12:58

1. a linear combination of $\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n\}$
   is any $\vec{v} = x_1\vec{v}_1 + x_2\vec{v}_2 + \ldots + x_n\vec{v}_n$ with $x_1, x_2 \ldots x_n \in \mathbb{R}$

2. span $\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n\}$
   = collection of all linear combinations

   geometrically span$\{\vec{v}_1\}$ is line thru $\vec{0}$. ($\vec{v}_1 \neq \vec{0}$)
   span$\{\vec{v}_1, \vec{v}_2\}$ is a plane thru $\vec{0}$ (as long as $\vec{v}_1, \vec{v}_2$ not parallel)

3. $A\vec{x}$, for $A_{\underline{m \times n}}$ & $\vec{x} \in \mathbb{R}^n$

$$\left[\vec{a}_1 \middle| \vec{a}_2 \middle| \cdots \vec{a}_n\right]\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\vec{a} + x_2\vec{a}_2 + \ldots + x_n\vec{a}_n$$

(also $\begin{bmatrix} Row_1(A) \\ \hline Row_2(A) \\ \hline \vdots \\ Row_m(A) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} Row_1(A)\cdot\vec{x} \\ Row_2(A)\cdot\vec{x} \\ \vdots \\ Row_m(A)\cdot\vec{x} \end{bmatrix}$)

4. Homogeneous matrix equation
   $A\vec{x} = \vec{0}$
   Nonhomogeneous " "
   $A\vec{x} = \vec{b}$, $\vec{b} \neq \vec{0}$.

5. Pivot location for a matrix B
   location of a pivot in r.r.e.f.(B).

6. 4 conditions for a matrix to be in reduced row echelon form
   (1) all zero rows are at bottom
   (2) pivots (the 1st non-zero entry in a row, move to the right as you descend rows
   (3) pivots = 1
   (4) entries in any pivot column = 0, except for pivot

Review and consolidation of facts from sections 1.1-1.5:

1) If $A_{m \times n} = [\underline{\boldsymbol{a}}_1\ \underline{\boldsymbol{a}}_2\ \ldots\ \underline{\boldsymbol{a}}_n]$ is expressed in terms of its columns, with a_{ij} being the i^{th} entry of $\underline{\boldsymbol{a}}_j$ then we know

$$A\,\underline{\boldsymbol{x}} := x_1\underline{\boldsymbol{a}}_1 + x_2\,\underline{\boldsymbol{a}}_2 + \ldots + x_n\,\underline{\boldsymbol{a}}_n = \begin{bmatrix} x_1a_{11} + x_2a_{12} + \ldots + x_na_{1n} \\ x_1a_{21} + x_2a_{22} + \ldots + x_na_{2n} \\ \vdots \\ x_1a_{ml} + x_2a_{m2} + \ldots + x_na_{mn} \end{bmatrix} = \begin{bmatrix} Row_1(A)\cdot\underline{\boldsymbol{x}} \\ Row_2(A)\cdot\underline{\boldsymbol{x}} \\ \vdots \\ Row_m(A)\cdot\underline{\boldsymbol{x}} \end{bmatrix}.$$

So the matrix equation

$$A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}$$

from 1.4 represents

1a) systems of linear equations, as in 1.1-1.2, as well as

1b) vector (linear combination) equations, as in section 1.3.

The solution set in any such problem is found and understood by reducing the augmented matrix $[\underline{\boldsymbol{a}}_1, \underline{\boldsymbol{a}}_2, \ldots\ \underline{\boldsymbol{a}}_n, \underline{\boldsymbol{b}}]$ to see if the system is consistent, and then backsolving when it is.

2) We can understand a lot about the geometry of the solution set of the matrix equation $A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}$ based on the shape of the reduced row echelon form of the augmented matrix

$$[A, \underline{\boldsymbol{b}}] = \left[\underline{\boldsymbol{a}}_1, \underline{\boldsymbol{a}}_2, \ldots \underline{\boldsymbol{a}}_n, \underline{\boldsymbol{b}}\right],$$

or often just on the shape of the reduced row echelon form of

$$A = \left[\underline{\boldsymbol{a}}_1, \underline{\boldsymbol{a}}_2, \ldots \underline{\boldsymbol{a}}_n\right]$$

alone.

2a) The system is inconsistent if and only if what is true about $rref([A, \underline{\boldsymbol{b}}])$?

$\exists$ row "0" on left, 1 on right

i.e. right-most column has a pivot (eqtn says $0=1$)

row: $0\ 0\ 0\ 0 \,\vdots\, 1$

2b) If the system is consistent then there is a unique (exactly one) solution $\underline{\boldsymbol{x}}$ to $A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}$ if and only if what is true about $rref(A)$?

every column has a pivot

$$\left[\begin{array}{cc|c} 1 & 0 & c_1 \\ 0 & 1 & c_2 \\ 0 & 0 & 0 \end{array}\right] \quad \text{unique}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & c_1 \\ 0 & 0 & 1 & c_2 \end{array}\right] \quad \text{not unique.}$$

2c) If the system is consistent then the number of free variables in the solution is given by what number related to $rref(A)$?

# of cols of rref(A) without pivots.

2d) For a fixed matrix A the matrix equation $A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}$ is consistent for all possible choices of $\underline{\boldsymbol{b}}$ if and only if what is true about $rref(A)$?

no zero rows in rref(A)

i.e. every row in rref(A) has a pivot.

because I can solve for the pivot variable in terms of the later variables (that shows $A\vec{x} = \vec{b}$ is always consistent in this case).

e.g.

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & d_1 \\ 0 & 1 & 2 & d_2 \end{array}\right]$$

$A\vec{x} = \vec{b}$

Note: if instead, rref(A) has a row of zeroes: $A \xrightarrow{rref} \begin{matrix} \ldots \\ 0\,0\,0\,0 \end{matrix}$

then there are inconsistent systems $A\vec{x} = \vec{b}$: $A \,\vdots\, \vec{b} \xleftarrow{rref^{-1}} \begin{matrix} \ldots & \vdots \\ 0\,0\,0\,0 & \vdots\, 1 \end{matrix}$

3) Let $A_{n \times n}$ be a square matrix.

3a) Then the matrix equation $A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}$ is consistent for all possible choices of $\underline{\boldsymbol{b}}$ if and only if what is true about $rref(A)$?

every row of rref(A) must have a pivot (=1), so n pivots

i.e. $rref(A) = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots \\ \vdots & & \ddots & \\ 0 & 0 & \cdots 0 & 1 \end{bmatrix}$

$:= I$

3b) Then solutions to the matrix equation $A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}$ are unique if and only if what is true about $rref(A)$?

every column of rref(A) must have a pivot, i.e. n pivots

i.e. $rref(A) = I$ also

4) spanning sets

4a) Fewer than m vectors in $\mathbb{R}^m$, $\{\underline{\boldsymbol{a}}_1, \underline{\boldsymbol{a}}_2, \ldots \underline{\boldsymbol{a}}_n\}$ with $n < m$, will never span all of $\mathbb{R}^m$ because

2 vectors in $\mathbb{R}^3$: $x_1\vec{a}_1 + x_2\vec{a}_2 = \vec{b}$ solvable for all $\vec{b} \in \mathbb{R}^3$?

$A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 \end{bmatrix}_{3\times 2} \xrightarrow{rref} \begin{bmatrix} \ast & \ast \\ \ast & \ast \\ 0 & 0 \end{bmatrix}$

so can't always solve $A\vec{x} = \vec{b}$.

(this same reasoning holds whenever $n<m$.)

4b) Exactly n vectors in $\mathbb{R}^n$, $\{\underline{\boldsymbol{a}}_1, \underline{\boldsymbol{a}}_2, \ldots \underline{\boldsymbol{a}}_n\}$ span $\mathbb{R}^n$ if and only if

i.e. can always solve $A\vec{x} = \vec{b}$ for $\vec{x}$

every row has a pivot, i.e. as in 3a),

$rref(A) = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} = I$ (as in 3a)