Fri Jan 19

- 1.5 solution sets to matrix equations; homogeneous and non-homogeneous systems of equations.

Announcements:

- Canvas exists
- Posted all of Chapter 1 on Canvas
  (I'll post rest of problem sets).
- FFT in 2^{nd} half of class.

~~Warm up Exercise~~:

recall:

$A\vec{x} = \vec{b}$ is shorthand for

$$x_1\vec{a}_1 + x_2\vec{a}_2 + \ldots + x_n\vec{a}_n = \vec{b}$$

(where $A = \left[\vec{a}_1 \middle| \vec{a}_2 \middle| \cdots \vec{a}_n\right]$).

also shorthand for linear system with augmented matrix

$$A \,\vdots\, \vec{b}$$

Definition: A system of linear equations is *homogeneous* if it can be written in the form

$$A\,\underline{\mathbf{x}} = \underline{\mathbf{0}}$$

where A is an $m \times n$ matrix, and $\underline{\mathbf{0}}$ is the zero vector in $\mathbb{R}^m$.

Definition: A system of linear equations is *nonhomogeneous* if it can be written in the form

$$A\,\underline{\mathbf{x}} = \underline{\mathbf{b}}$$

where A is an $m \times n$ matrix, and $\underline{\mathbf{b}}$ is non-zero, i.e. not the zero vector in $\mathbb{R}^m$.

Our goal in section 1.5 is to understand the relationship between the solution sets of homogeneous and nonhomogeneous sytems, when the matrix A is the same.

To understand how the different solution sets are related, we will check and use these algebra facts:

$$A\,(\underline{\mathbf{x}} + \underline{\mathbf{y}}) = A\,\underline{\mathbf{x}} + A\,\underline{\mathbf{y}}$$

$$A\,(c\,\underline{\mathbf{x}}) = c\,A\,\underline{\mathbf{x}}\,.$$

$$\left[\vec{a}_1 \,\middle|\, \vec{a}_2 \,\middle|\, \cdots \,\middle|\, \vec{a}_n\right] \begin{bmatrix} x_1+y_1 \\ x_2+y_2 \\ \vdots \\ x_n+y_n \end{bmatrix} = (x_1+y_1)\vec{a}_1 + (x_2+y_2)\vec{a}_2 + \cdots + (x_n+y_n)\vec{a}_n$$

$$A x + A\vec{y}.$$

$$\left[\vec{a}_1 \,\middle|\, \vec{a}_2 \,\middle|\, \cdots \,\middle|\, \vec{a}_n\right] \begin{bmatrix} cx_1 \\ cx_2 \\ \vdots \\ cx_n \end{bmatrix} = cx_1\vec{a}_1 + cx_2\vec{a}_2 + \cdots + cx_n\vec{a}_n$$

$$= c\left(A\vec{x}\right)$$

Warm-up exercise : until 1:00

<u>Homogeneous systems</u>: Notice that for any matrix A, it's always true that the homogeneous equation $A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{0}}$ has a solution $\underline{\boldsymbol{x}} = \underline{\boldsymbol{0}}$, so homogeneous systems are always consistent. The question is whether there are more solutions. (And, we call the solution $\underline{\boldsymbol{x}} = \underline{\boldsymbol{0}}$ the "trivial" solution.)

<u>Exercise 1)</u> Find and compare the solution sets of the following two linear systems. The first one is homogeneous and the second one is non-homogeneous. How do the solutions sets appear to be related?

1

$$\begin{aligned} 3x_1 + 5x_2 - 4x_3 &= 0 \\ -3x_1 - 2x_2 + 4x_3 &= 0 \\ 6x_1 + x_2 - 8x_3 &= 0 \end{aligned}$$

2

$$\begin{aligned} 3x_1 + 5x_2 - 4x_3 &= 7 \\ -3x_1 - 2x_2 + 4x_3 &= -1 \\ 6x_1 + x_2 - 8x_3 &= -4 \end{aligned}$$

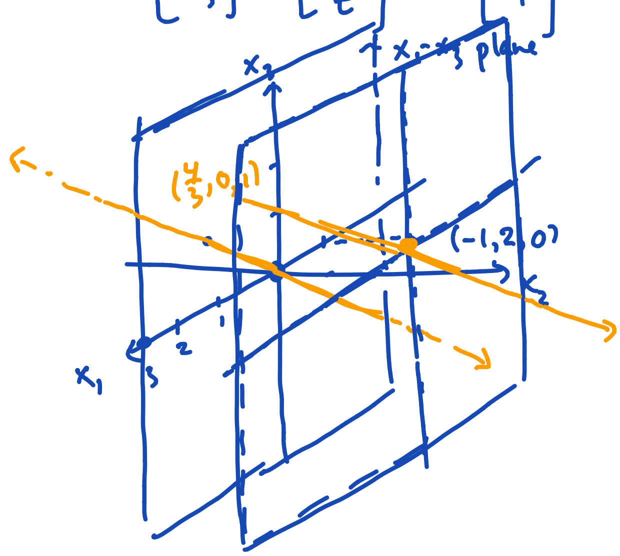
$$\left[\begin{array}{ccc|cc} 3 & 5 & -4 & 0 & 7 \\ -3 & -2 & 4 & 0 & -1 \\ 6 & 1 & -8 & 0 & -4 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|cc} 1 & 0 & -4/3 & 0 & -1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

1 soltn $\quad x_1 = \frac{4}{3}x_3 = \frac{4}{3}t$
$x_2 = 0$
$x_3 = \text{free} = t$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{4}{3}t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}$$

2 $\quad x_1 = -1 + \frac{4}{3}t$
$x_2 = 2$
$x_3 = \text{free} = t$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 + \frac{4}{3}t \\ 2 \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{4}{3}t \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}$$



each solution gives position vectors of points on parallel lines
the second line is the first line, translated by the soltn $\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$ to 2nd system.

What happened in Exercise 1 is what always happens when the non-homogeneous system is consistent. It says that for consistent nonhomogeneous systems, all solution sets are "translations" of each other.

<u>Theorem</u> (Fundamental Theorem of matrix equations) Suppose the equation $\boxed{A\,\underline{\mathbf{x}} = \underline{\boldsymbol{b}}}$ is consistent for some $\underline{\boldsymbol{b}}$. Let $\underline{\boldsymbol{p}}$ be a solution. Then the solution set of $A\,\underline{\mathbf{x}} = \underline{\boldsymbol{b}}$ is the set of all vectors

$$\underline{\boldsymbol{w}} = \underline{\boldsymbol{p}} + \underline{\boldsymbol{v}}_h$$

where $\underline{\boldsymbol{v}}_h$ is any solution of the homogeneous equation

$\boxed{A\,\underline{\mathbf{x}} = \underline{\boldsymbol{0}}.}$ ← is always consistent, ($\vec{x} = \vec{0}$ solves homog eqn).

We can verify why this theorem is true!

If $\vec{p}$ ("particular solution) solves $A\vec{x} = \vec{b}$.
Then the solution set to $A\vec{x} = \vec{b}$ is
$\vec{x} = \vec{p} + \vec{v}_h$ where $\vec{v}_h$ is any (i.e. all) soln to homog problem.

<u>proof</u> : (1) know $A\vec{p} = \vec{b}$.
Let $A\vec{v}_h = \vec{0}$.
then $A(\vec{p} + \vec{v}_h) = A\vec{p} + A\vec{v}_h$
$= \vec{b} + \vec{0} = \vec{b}$.

(2) If $A\vec{q} = \vec{b}$
then $\vec{q} = \vec{p} + \underbrace{(\vec{q} - \vec{p})}$
$A(\vec{q} - \vec{p}) = A\vec{q} - A\vec{p} = \vec{b} - \vec{b} = \vec{0}$.
So $\vec{q} = \vec{p} + \vec{v}_H$ (where $\vec{v}_H = \vec{q} - \vec{p}$).