Fri Jan 19

- 1.5 solution sets to matrix equations; homogeneous and non-homogeneous systems of equations.

**Announcements:**

- Canvas exists
- Posted all of Chpter 1 on Canvas (I’ll post rest of problem sets).
- FFT in 2nd half of class.

**Warm-up Exercise:**

Recall:

\[ A \vec{x} = \vec{b} \]

is shorthand for

\[ x_1 \vec{a}_1 + x_2 \vec{a}_2 + \ldots + x_n \vec{a}_n = \vec{b} \]

(where \( A = [\vec{a}_1 | \vec{a}_2 | \ldots | \vec{a}_n] \)).

Also shorthand for linear system with augmented matrix

\[ \begin{bmatrix} A & \mid & \vec{b} \end{bmatrix} \]
Definition: A system of linear equations is **homogeneous** if it can be written in the form

\[ A \mathbf{x} = \mathbf{0} \]

where \( A \) is an \( m \times n \) matrix, and \( \mathbf{0} \) is the zero vector in \( \mathbb{R}^m \).

Definition: A system of linear equations is **nonhomogeneous** if it can be written in the form

\[ A \mathbf{x} = \mathbf{b} \]

where \( A \) is an \( m \times n \) matrix, and \( \mathbf{b} \) is non-zero, i.e. not the zero vector in \( \mathbb{R}^m \).

Our goal in section 1.5 is to understand the relationship between the solution sets of homogeneous and nonhomogeneous systems, when the matrix \( A \) is the same.

To understand how the different solution sets are related, we will check and use these algebra facts:

\[ A (\mathbf{x} + \mathbf{y}) = A \mathbf{x} + A \mathbf{y} \]

\[ A (c \mathbf{x}) = c A \mathbf{x} \]

\[
\begin{bmatrix}
\mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_n
\end{bmatrix}
\begin{bmatrix}
x_1 + y_1 \\
x_2 + y_2 \\
\vdots \\
x_n + y_n
\end{bmatrix}
= (x_1 + y_1) \mathbf{a}_1 + (x_2 + y_2) \mathbf{a}_2 + \ldots + (x_n + y_n) \mathbf{a}_n
\]

\[
\begin{bmatrix}
\mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_n
\end{bmatrix}
\begin{bmatrix}
c_1 x_1 \\
c_2 x_2 \\
\vdots \\
c_n x_n
\end{bmatrix}
= c_1 \mathbf{a}_1 + c_2 \mathbf{a}_2 + \ldots + c_n \mathbf{a}_n
\]

\[ = c \left( A \mathbf{x} \right) \]
Homogeneous systems: Notice that for any matrix $A$, it's always true that the homogeneous equation $A \mathbf{x} = \mathbf{0}$ has a solution $\mathbf{x} = \mathbf{0}$, so homogeneous systems are always consistent. The question is whether there are more solutions. (And, we call the solution $\mathbf{x} = \mathbf{0}$ the "trivial" solution.)

Exercise 1) Find and compare the solution sets of the following two linear systems. The first one is homogeneous and the second one is non-homogeneous. How do the solutions sets appear to be related?

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1. \[
\begin{align*}
3x_1 + 5x_2 - 4x_3 &= 0 \\
-3x_1 - 2x_2 + 4x_3 &= 0 \\
6x_1 + x_2 - 8x_3 &= 0
\end{align*}
\]

2. \[
\begin{align*}
3x_1 + 5x_2 - 4x_3 &= 7 \\
-3x_1 - 2x_2 + 4x_3 &= -1 \\
6x_1 + x_2 - 8x_3 &= -4
\end{align*}
\]

Solution 1:
\[
\begin{align*}
x_1 &= \frac{4}{3}x_3 + \frac{1}{3}t \\
x_2 &= 0 \\
x_3 &= \text{free} = t
\end{align*}
\]

Solution 2:
\[
\begin{align*}
x_1 &= -1 + \frac{4}{3}t \\
x_2 &= 2 \\
x_3 &= \text{free} = t
\end{align*}
\]

Each solution gives position vectors of points on parallel lines; the second line is the first line, translated by the solution to 2nd system.

Warm-up exercise: until 1:00
What happened in Exercise 1 is what always happens when the non-homogeneous system is consistent. It says that for consistent nonhomogeneous systems, all solution sets are "translations" of each other.

**Theorem (Fundamental Theorem of matrix equations)** Suppose the equation $Ax = b$ is consistent for some $b$. Let $p$ be a solution. Then the solution set of $Ax = b$ is the set of all vectors

$$w = p + v_h$$

where $v_h$ is any solution of the homogeneous equation

$$Ax = 0.$$  

(\text{is always consistent, } \{x = 0 \text{ solves homog eqn}\})

We can verify why this theorem is true!

**proof:**

1. **know $Ap = b.$**
   
   Let $Av_h = 0.$
   
   then $A(p + v_h) = Ap + Av_h = b + 0 = b.$

2. **If $Aq = b.$**
   
   then $q = p + q - p = p + \frac{q - p}{A}$
   
   so $q = p + v_h \text{ (where } v_h = q - p).$