What we may have realized in the previous exercise is the very important:

Fundamental Fact A *vector equation* (linear combination problem)

$$x_1\,\underline{\boldsymbol{a}}_1 + x_2\,\underline{\boldsymbol{a}}_2 + \ldots + x_n\,\underline{\boldsymbol{a}}_n = \underline{\boldsymbol{b}}$$

is actually a system of linear equations for the unknown weights $x_1, x_2, \ldots\, x_n$; in fact the system of linear equations has augmented matrix given by

$$\left[\underline{\boldsymbol{a}}_1\ \ \underline{\boldsymbol{a}}_2\ \cdots\ \underline{\boldsymbol{a}}_n\ \ \underline{\boldsymbol{b}}\right]$$

(where we have expressed the augmented matrix in terms of its columns). In particular, $\underline{\boldsymbol{b}}$ can be generated by a linear combination of $\underline{\boldsymbol{a}}_1\ \ \underline{\boldsymbol{a}}_2\ \ldots\ \underline{\boldsymbol{a}}_n$ if and only if there exists a solution to the linear system corresponding to the augmented matrix above.

This fundamental fact is so important to the course, that we should check it in general at some point.

$$\underbrace{x_1\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1}\end{bmatrix}}_{\vec{a}_1} + \underbrace{x_2\begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2}\end{bmatrix}}_{\vec{a}_2} + \ldots + \underbrace{x_n\begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{nn}\end{bmatrix}}_{\vec{a}_n} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n\end{bmatrix}$$

Exercise 3a) Does the vector equation

$$x_1\begin{bmatrix} 1 \\ 0 \\ 2\end{bmatrix} + x_1\begin{bmatrix} -1 \\ 2 \\ 0\end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 1\end{bmatrix}$$

have any solutions?

you just did this in warmup.

yes.. $x_1 = \frac{1}{2}$

$x_2 - \frac{3}{2}$

3b) What geometric question is this related to? What geometric object is $span\left\{\begin{bmatrix} 1 \\ 0 \\ 2\end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0\end{bmatrix}\right\}$?

this will be a plane in $\mathbb{R}^3$.

is $\begin{bmatrix} 2 \\ -3 \\ 1\end{bmatrix}$ in the plane spanned by $\begin{bmatrix} 1 \\ 0 \\ 2\end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0\end{bmatrix}$?

3c) Use an augmented matrix calculation to find what condition needs to hold on vectors $\underline{\boldsymbol{b}}$ so that

$$\underline{\boldsymbol{b}} \in span\left\{\begin{bmatrix}1\\0\\2\end{bmatrix}, \begin{bmatrix}-1\\2\\0\end{bmatrix}\right\}. \quad (!!)$$

$$x_1\begin{bmatrix}1\\0\\2\end{bmatrix} + x_2\begin{bmatrix}-1\\2\\0\end{bmatrix} = \begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$$

which $\vec{b}$?

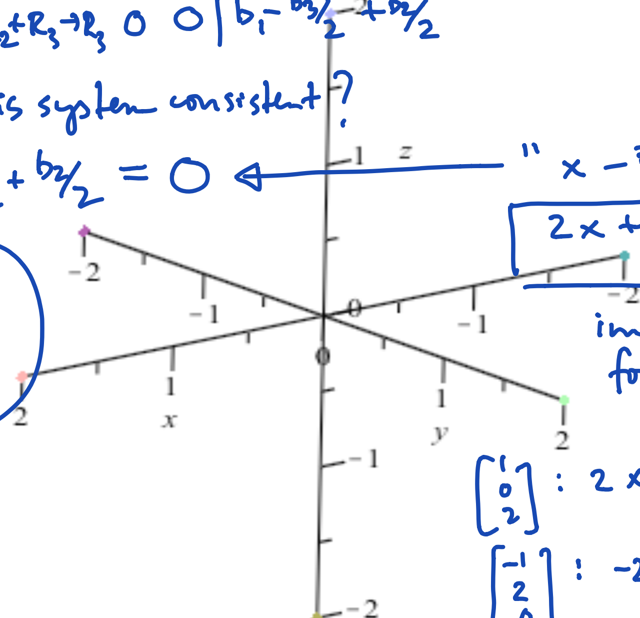
$$\left[\begin{array}{cc|c} 1 & -1 & b_1 \\ 0 & 2 & b_2 \\ 2 & 0 & b_3 \end{array}\right]$$

$$\begin{array}{l} R_3/2 \to R_1 \\ R_2/2 \to R_2 \\ R_1 \to R_3 \end{array} \quad \left[\begin{array}{cc|c} 1 & 0 & b_3/2 \\ 0 & 1 & b_2/2 \\ 1 & -1 & b_1 \end{array}\right]$$

$$-R_1 + R_3 \to R_1 \quad \left[\begin{array}{cc|c} 1 & 0 & b_3/2 \\ 0 & 1 & b_2/2 \\ 0 & -1 & -b_3/2 + b_1 \end{array}\right]$$

In case we want to sketch anything related to Exercise 3:

$$R_2 + R_3 \to R_3 \quad \left[\begin{array}{cc|c} 1 & 0 & b_3/2 \\ 0 & 1 & b_2/2 \\ 0 & 0 & b_1 - b_3/2 + b_2/2 \end{array}\right]$$



Q: when is this system consistent?

A: $b_1 - b_3/2 + b_2/2 = 0$ ← " $x - z/2 + y/2 = 0$

$\boxed{2x + y - z = 0}$

implicit eqtn for the plane

& then

$x_1 = b_3/2$

$x_2 = b_2/2$

$\begin{bmatrix}1\\0\\2\end{bmatrix}$: $2x + y - z = 2 - 2 = 0$ ✓

$\begin{bmatrix}-1\\2\\0\end{bmatrix}$: $-2 + 2 = 0$ ✓

$\begin{bmatrix}2\\-3\\1\end{bmatrix}$: $4 - 3 - 1 = 0$ ✓

& $\begin{bmatrix}2\\-3\\1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\0\\2\end{bmatrix} - \frac{3}{2}\begin{bmatrix}-1\\2\\0\end{bmatrix}$

Wed Jan 17

- 1.4 the matrix equation $A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}$.

Announcements:
- 1.4 11, 13, 17-25 odd, 25, 31 should be doable after today (part of next week's HW.)
- quiz
- finish T notes & today's.

Warm-up Exercise: Is $\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$ in the span of $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \right\}$?

span: is the collection of all linear combinations of the (two) vectors in the set

linear combinations: sum of scalar multiples

hint: the vector equation you're trying to solve is

§1.3 vector eqtn.

$$* \quad x_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$$

'til 1:03

answer: YES.

$x_1 = \frac{1}{2}$

$x_2 = -\frac{3}{2}$

check: $\frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - \frac{3}{2} \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 3/2 \\ -3 \\ 0 \end{bmatrix} \checkmark = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$$

since we found a linear combination of $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$ that equals $\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$ is in span $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \right\}$

method: * has augmented matrix

$$\left[\begin{array}{cc|c} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 2 & 0 & 1 \end{array}\right]$$

reduce! backsolve if system is consistent.

Recall

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is actually a system of linear equations for the unknown weights $x_1, x_2, \ldots. x_n$; in fact the system of linear equations has augmented matrix given by

$$\left[\underline{\boldsymbol{a}}_1 \;\; \underline{\boldsymbol{a}}_2 \;\; \cdots \;\; \underline{\boldsymbol{a}}_n \;\; \underline{\boldsymbol{b}}\,\right]$$

(where we have expressed the augmented matrix in terms of its columns). In particular, $\underline{\boldsymbol{b}}$ can be generated by a linear combination of $\underline{\boldsymbol{a}}_1 \;\; \underline{\boldsymbol{a}}_2 \;\; \ldots \;\; \underline{\boldsymbol{a}}_n$ if and only if there exists a solution to the linear system corresponding to the augmented matrix above.
We should check this carefully today, assuming we didn't do so on Tuesday: ✓

Definition (from 1.4) If A is an $m \times n$ matrix, with columns $\underline{\boldsymbol{a}}_1, \underline{\boldsymbol{a}}_2, \ldots \underline{\boldsymbol{a}}_n$ (in $\mathbb{R}^m$) and if $\underline{\boldsymbol{x}} \in \mathbb{R}^n$, then $A\,\underline{\boldsymbol{x}}$ is defined to be the linear combination of the columns, with weights given by the corresponding entries of $\underline{\boldsymbol{x}}$. In other words,

$$A\,\underline{\boldsymbol{x}} := x_1\,\underline{\boldsymbol{a}}_1 + x_2\,\underline{\boldsymbol{a}}_2 + \ldots x_n\,\underline{\boldsymbol{a}}_n\,.$$

(This will give us a way to abbreviate vector equations.)

<u>Definition.</u> Let $\underline{u}$, $\underline{v}$ be vectors in $\mathbb{R}^n$. Then the *dot product* $\underline{u} \cdot \underline{v}$ is defined by

$$\underline{u} \cdot \underline{v} = \sum_{j=1}^{n} u_j v_j = u_1 v_1 + u_2 v_2 + \dots u_n v_n .$$

<u>Computational Theorem:</u> (This is usually a quicker way to compute $A\,\underline{x}$. Let If A be an $m \times n$ matrix, with rows $R_1, R_2, \dots R_m$. Then $A\,\underline{x}$ may also be computed using the rows of A and the dot product:

$$x_1 \underline{a}_1 + x_2 \underline{a}_2 + \dots x_n \underline{a}_n = A\,\underline{x} = \begin{bmatrix} R_1 \cdot \underline{x} \\ R_2 \cdot \underline{x} \\ \vdots \\ R_m \cdot \underline{x} \end{bmatrix}$$

<u>Exercise 1a)</u> Compute both ways:

$$\begin{array}{c} R_1 \rightarrow \\ R_2 \rightarrow \end{array} \begin{bmatrix} 1 & -2 & 3 \\ -2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = 2\begin{bmatrix} 1 \\ -2 \end{bmatrix} - 2\begin{bmatrix} -2 \\ 3 \end{bmatrix} + 1\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2+4+3 \\ -4-6+4 \end{bmatrix} = \begin{bmatrix} 9 \\ -6 \end{bmatrix}$$

$$= \begin{bmatrix} R_1 \cdot \vec{x} \\ R_2 \cdot \vec{x} \end{bmatrix} = \begin{bmatrix} 1\cdot 2 + (-2)(-2) + 3\cdot 1 \\ (-2)(2) + 3(-2) + 4\cdot 1 \end{bmatrix} = \begin{bmatrix} 2+4+3 \\ -4-6+4 \end{bmatrix}$$

<u>Exercise 1b)</u> Write as a matrix times a vector:

$$3\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} + 2\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} =$$