Math 2270-004 Week 2 notes

We will not necessarily finish the material from a given day's notes on that day. We may also add or subtract some material as the week progresses, but these notes represent an in-depth outline of what we plan to cover. These notes cover material in 1.3-1.5. They include material from last weeks notes that we did not get to.

Tues Jan 16

• 1.3 algebra and geometry for vector equations and linear combinations

Warm-up Exercise:
$$f$$
 lere's the reduced row echelon form
of an anguented matrix.
 $O(D O 3 | 5$
 $O O (D -7 | 2)$
 $X_1 = free!$
 $X_2 = 5 - 3x_4$
 $X_3 = 2 + 7x_4$
 $X_4 = free$
 $X_3 = 2 + 7x_4$
 $X_4 = free$

On Friday we defined vectors algebraically, as ordered lists of numbers. And, we defined vector addition and scalar multiplication:

$$\underline{\text{Definition: For } \boldsymbol{\mu}} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \quad \boldsymbol{\nu} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n; \quad \boldsymbol{c} \in \mathbb{R}, \text{ then } \quad \boldsymbol{\mu} + \boldsymbol{\nu} \coloneqq \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix}; \quad \boldsymbol{c} \, \boldsymbol{\mu} \coloneqq \begin{bmatrix} cu_1 \\ cu_2 \\ \vdots \\ cu_n \end{bmatrix}.$$

There are a number of straightforward algebra properties for vector addition and scalar multiplication:

Let $\underline{u}, \underline{v}, \underline{w} \in \mathbb{R}^n$, $c, d \in \mathbb{R}$. Then

- (i) $\underline{u} + \underline{v} = \underline{v} + \underline{u}$
- (ii) $(\underline{u} + \underline{v}) + \underline{w} = \underline{u} + (\underline{v} + \underline{w})$
- (iii) $\underline{u} + \underline{0} = \underline{0} + \underline{u} = \underline{u}$ ($\underline{0}$ is defined to be the vector for which each entry is the number 0.)

(iv) $\underline{u} + (-\underline{u}) = (-\underline{u}) + \underline{u} = \underline{0} (-\underline{u}$ is defined to be $-1 \cdot \underline{u}$, i.e. the vector for which each entry is the opposite of the corresponding entry in \underline{u} .)

- (v) $c(\underline{u} + \underline{v}) = c \, \underline{u} + c \, \underline{v}$
- (vi) $(c+d)\underline{u} = c \,\underline{u} + d \,\underline{u}$

(vii) $c(d \underline{u}) = (c d) \underline{u}$

(viii) $1 \underline{u} = \underline{u}$.

Geometric interpretation of vectors

The space \mathbb{R}^n may be thought of in two equivalent ways. In both cases, \mathbb{R}^n consists of all possible n - tuples of numbers:

(i) We can think of those n - tuples as representing points, as we're used to doing for n = 1, 2, 3. In this case we can write

$$\mathbb{R}^{n} = \left\{ \left(x_{1}, x_{2}, ..., x_{n} \right), s.t. x_{1}, x_{2}, ..., x_{n} \in \mathbb{R} \right\}.$$

(ii) We can think of those n - tuples as representing vectors that we can add and scalar multiply. In this case we can write

$$\mathbb{R}^{n} = \left\{ \begin{bmatrix} x_{1} \\ x_{2} \\ \vdots \\ x_{n} \end{bmatrix}, s.t. x_{1}, x_{2}, \dots, x_{n} \in \mathbb{R} \right\}.$$

Since algebraic vectors (as above) can be used to measure geometric displacement, one can identify the two models of \mathbb{R}^n as sets by identifying each point (x_1, x_2, \dots, x_n) in the first model with the displacement vector $\mathbf{x} = [x_1, x_2, \dots, x_m]^T$ from the origin to that point, in the second model, i.e. the "position vector" of the point.



Exercise 1) Let
$$\underline{\boldsymbol{u}} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$
 and $\underline{\boldsymbol{v}} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

<u>1a</u>) Plot the points (1,-1) and (1,3), which have position vectors $\underline{u}, \underline{v}$. Draw these position vectors as arrows beginning at the origin and ending at the corresponding points.

<u>1b</u>) Compute $\underline{u} + \underline{v}$ and then plot the point for which this is the position vector. Note that the algebraic operation of vector addition corresponds to the geometric process of composing horizontal and vertical displacements.

$$\vec{n} + \vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

<u>1c</u>) Compute 3 \underline{u} and $-2 \underline{v}$ and plot the corresponding points for which these are the position vectors.



One of the key themes of this course is the idea of "linear combinations". These have an algebraic definition, as well as a geometric interpretation as combinations of displacements.

<u>Definition</u>: If we have a collection of *n* vectors $\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_p\}$ in \mathbb{R}^n , then any vector $\underline{v} \in \mathbb{R}^n$ that can be expressed as a sum of scalar multiples of these vectors is called a *linear combination* of them. In other

then $\underline{\mathbf{v}}$ is a linear combination of $\underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2, \dots, \underline{\mathbf{v}}_p$. The scalars c_1, c_2, \dots, c_p are called the linear combination of coefficients or weights. The scalars $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$. The scalars c_1, c_2, \dots, c_p are called the linear combination

Example You've probably seen linear combinations in previous math/physics classes, even if you didn't realize it. For example you might have expressed the position vector \mathbf{r} as a linear combination

$$\underline{\mathbf{r}} = x\,\underline{\mathbf{i}} + y\,\underline{\mathbf{j}} + z\,\underline{\mathbf{k}}$$

where $\underline{i}, \underline{j}, \underline{k}$ represent the unit displacements in the x, y, z directions. Since we can express these displacements using Math 2270 notation as

	1		0		0]
<u>i</u> =	0	, j =	1	, <u>k</u> =	0	
	0		0		1	

we have

$$x \mathbf{i} + y \mathbf{j} + z \mathbf{k} = x \begin{bmatrix} 1\\0\\0 \end{bmatrix} + y \begin{bmatrix} 0\\1\\0 \end{bmatrix} + z \begin{bmatrix} 0\\0\\1 \end{bmatrix} = \begin{bmatrix} x\\y\\z \end{bmatrix}.$$

Exercise 2) Can you get to the point $(-2, 8) \in \mathbb{R}^2$, from the origin (0, 0), by moving only in the (\pm) directions of $\underline{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\underline{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$? Algebraically, this means we want to solve the linear combination problem

$$x_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 8 \end{bmatrix}.$$

<u>2a</u>) Superimpose a grid related to the displacement vectors \underline{u} , \underline{v} onto the graph paper below, and, recalling that vector addition yields net displacement, and scalar multiplication yields scaled displacement, try to approximately solve the linear combination problem above, geometrically.

<u>2b)</u> Rewrite the linear combination problem as a linear system and solve it exactly, algebraically!!

$$2a): \begin{bmatrix} -2\\ 8 \end{bmatrix} \approx -3.5 \vec{u} + 1.5 \vec{v}$$

$$2b) \times_{1} \begin{bmatrix} 1\\ -1 \end{bmatrix} + \times_{2} \begin{bmatrix} 1\\ 3 \end{bmatrix} = \begin{bmatrix} -2\\ 8 \end{bmatrix}$$

$$\begin{bmatrix} x_{1} + x_{2} \\ -x_{1} + 3x_{2} \end{bmatrix} = \begin{bmatrix} -2\\ 8 \end{bmatrix}$$

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$$x_{1} + x_{2} = -2$$

$$-x_{1} + 3x_{2} = 8$$

$$\begin{bmatrix} 1\\ 1\\ 1\\ -1 \end{bmatrix} = -2$$

$$R_{1} + R_{2} - 3R_{2} = \underbrace{O + \frac{1}{6} - \frac{1}{6}}_{1 - \frac{1}{2} - 2}$$

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$$R_{1} + R_{2} - 3R_{2} = \underbrace{O + \frac{1}{6} - \frac{1}{6}}_{1 - \frac{1}{2} - 2}$$

$$R_{1} + R_{2} - 3R_{3} = \underbrace{O + \frac{1}{1.5}}_{0 - \frac{1}{1.5} - 2} = 2 + 3.5 \\ x_{2} = 1.5 = 5 \circ \begin{bmatrix} -2\\ 8 \end{bmatrix} = -3.5 \begin{bmatrix} 1\\ -1 \end{bmatrix} + 1.5 \begin{bmatrix} 1\\ 3 \end{bmatrix}$$

$$= -3.5 \vec{u} + 1.5 \vec{v}$$

$$Exactly]$$

1

<u>2c)</u> Can you get to any point (x, y) in \mathbb{R}^2 , starting at (0, 0) and moving only in directions parallel to $\underline{u}, \underline{v}$? Argue geometrically and algebraically. How many ways are there to express $\begin{bmatrix} \mathbf{k}_1 \\ \mathbf{k}_2 \end{bmatrix}$ as a linear combination of \underline{u} and \underline{v} ?

yes: geometric reasoning : in and i point in different
(i.e. non-parallel) directions, so the grid they
generate covers all of IR²
algebraic reasoning is more precise:
the vector equation to solve is

$$x_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$
. from work or get
previous peak
This is a linear
system with augmented
matrix 1 1 ib reduce 1 0 ici so wighe solution
 $-1 3 i b_2 = 0 1 i c_2 i s_{1}^{-1}$

<u>Definition</u> The *span* of a collection of vectors, written as $span\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n\}$, is the collection of all linear $x_2 = c_2$ combinations of those vectors.

Examples: We showed in $\underline{2c}$ that $span\{\underline{u}, \underline{v}\} = \mathbb{R}^2$. On the other hand, $span\{\underline{u}\}$ is the line with implicit equation y = -x. **also**, $span\{\hat{c}, \hat{j}, \hat{h}\} = \mathbb{R}^3$, **from earlier in today's** <u>Remark</u>: The mathematical meaning of the word *span* is related to the English meaning - as in "wing

<u>Remark</u>: The mathematical meaning of the word *span* is related to the English meaning - as in "wing span" or "span of a bridge", but it's also different. The span of a collection of vectors goes on and on and does not "stop" at the vector or associated endpoint: