1.3 Vectors and vector equations: We'll carefully define vectors, algebraic operations on vectors and geometric interpretations of these operations, in terms of displacements. These ideas will eventually give us another way to interpret systems of linear equations.

Definition: A matrix with only one column, i.e an $n \times 1$ matrix, is a *vector* in $\mathbb{R}^n$. (We also call matrices with only one row "vectors", but in this section our vectors will always be column vectors.)

Examples:

2210 1320

$\underline{\boldsymbol{u}} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ is a vector in $\mathbb{R}^2$. $\langle 3, -1 \rangle$ $3\hat{i} - \hat{j}$ (same)

$\underline{\boldsymbol{v}} = \begin{bmatrix} 2 \\ -1 \\ 3 \\ 5 \\ 0 \end{bmatrix}$ is a vector in $\mathbb{R}^5$. ?

- We can multiply vectors by *scalars* (real numbers) and add together vectors that are the same size. We can combine these operations.

Example: For $\underline{\boldsymbol{u}} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ and $\underline{\boldsymbol{w}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ compute

$$\underline{\boldsymbol{u}} + 4\underline{\boldsymbol{w}} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} + \underbrace{4\begin{bmatrix} 1 \\ 2 \end{bmatrix}} = \begin{bmatrix} 7 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ -1 \end{bmatrix} + \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

Definition: For $\underline{\boldsymbol{u}} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \underline{\boldsymbol{v}} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n; c \in \mathbb{R}$,

$$\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}} := \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ : \\ u_n + v_n \end{bmatrix} \quad ; \quad c\,\underline{u} := \begin{bmatrix} cu_1 \\ cu_2 \\ : \\ cu_n \end{bmatrix}.$$

Exercise 1 Our vector notation may not be the same as what you used in math 2210 or Math 1320 or 1321. Let's discuss the notation you used, and how it corresponds to what we're doing here.

did

Vector addition and scalar multiplication have nice algebraic properties:

Let $\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}}, \underline{\boldsymbol{w}} \in \mathbb{R}^n$, $\quad c, d \in \mathbb{R}$. Then

(i) $\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}} = \underline{\boldsymbol{v}} + \underline{\boldsymbol{u}}$

(ii) $(\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}}) + \underline{\boldsymbol{w}} = \underline{\boldsymbol{u}} + (\underline{\boldsymbol{v}} + \underline{\boldsymbol{w}})$

(iii) $\underline{\boldsymbol{u}} + \underline{\mathbf{0}} = \underline{\mathbf{0}} + \underline{\boldsymbol{u}} = \underline{\boldsymbol{u}}$

(iv) $\underline{\boldsymbol{u}} + (-\underline{\boldsymbol{u}}) = (-\underline{\boldsymbol{u}}) + \underline{\boldsymbol{u}} = \underline{\mathbf{0}}$

(v) $c(\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}}) = c\,\underline{u} + c\,\underline{\boldsymbol{v}}$

(vi) $(c + d)\underline{\boldsymbol{u}} = c\,\underline{\boldsymbol{u}} + d\,\underline{\boldsymbol{u}}$

(vii) $c(d\,\underline{\boldsymbol{u}}) = (c\,d)\,\underline{\boldsymbol{u}}$

(viii) $1\,\underline{\boldsymbol{u}} = \underline{\boldsymbol{u}}$.

<u>Exercise 2</u>. Verify why these properties hold!

Geometric interpretation of vectors

The space $\mathbb{R}^n$ may be thought of in two equivalent ways. In both cases, $\mathbb{R}^n$ consists of all possible $n-$ *tuples* of numbers:

(i) We can think of those $n-$ *tuples* as representing points, as we're used to doing for $n = 1, 2, 3$. In this case we can write

$$\mathbb{R}^n = \{ (x_1, x_2, \ldots, x_n), s.t.\ x_1, x_2, \ldots, x_n \in \mathbb{R} \}.$$

(ii) We can think of those $n-$ *tuples* as representing vectors that we can add and scalar multiply. In this case we can write

$$\mathbb{R}^n = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ : \\ x_n \end{bmatrix}, s.t.\ x_1, x_2, \ldots, x_n \in \mathbb{R} \right\}.$$

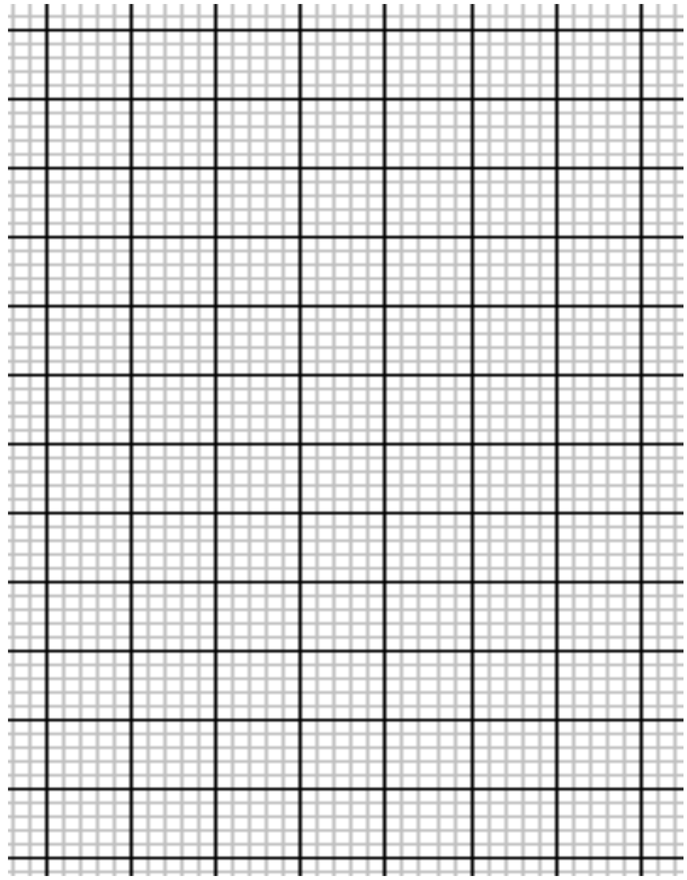
Since algebraic vectors (as above) can be used to measure geometric displacement, one can identify the two models of $\mathbb{R}^n$ as sets by identifying each point $(x_1, x_2, \ldots x_n)$ in the first model with the displacement vector $\underline{\boldsymbol{x}} = [x_1, x_2, \ldots x_m]^T$ from the origin to that point, in the second model, i.e. the position vector.

Exercise 3) Let $\underline{\boldsymbol{u}} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\underline{\boldsymbol{v}} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

1a) Plot the points $(1, -1)$ and $(1, 3)$, which have position vectors $\underline{\boldsymbol{u}}$, $\underline{\boldsymbol{v}}$. Draw these position vectors as arrows beginning at the origin and ending at the corresponding points.

1b) Compute $\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}}$ and then plot the point for which this is the position vector. Note that the algebraic operation of vector addition corresponds to the geometric process of composing horizontal and vertical displacements.

1c) Compute $3\,\underline{\boldsymbol{u}}$ and $-2\,\underline{\boldsymbol{v}}$ and plot the corresponding points for which these are the position vectors.



One of the key themes of this course is the idea of "linear combinations". These have an algebraic definition, as well as a geometric interpretation as combinations of displacements.

<u>Definition:</u> If we have a collection of n vectors $\{\underline{v}_1, \underline{v}_2, \ldots \underline{v}_p\}$ in $\mathbb{R}^n$, then any vector $\underline{v} \in \mathbb{R}^n$ that can be expressed as a sum of scalar multiples of these vectors is called a *linear combination* of them. In other words, if we can write

$$\underline{v} = c_1\underline{v}_1 + c_2\underline{v}_2 + \ldots + c_p\underline{v}_p ,$$

then $\underline{v}$ is a *linear combination* of $\underline{v}_1, \underline{v}_2, \ldots \underline{v}_p$. The scalars $c_1, c_2, \ldots, c_p$ are called the *linear combination coefficients* or *weights*.

<u>Example</u> You've probably seen linear combinations in previous math/physics classes, even if you didn't realize it. For example you might have expressed the position vector $\underline{\mathbf{r}}$ as a linear combination

$$\underline{\mathbf{r}} = x\,\underline{\mathbf{i}} + y\,\underline{\mathbf{j}} + z\,\underline{\mathbf{k}}$$

where $\underline{\mathbf{i}}, \underline{\mathbf{j}}, \underline{\mathbf{k}}$ represent the unit displacements in the x, y, z directions. Since we can express these displacements using Math 2270 notation as

$$\underline{\mathbf{i}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ \underline{\mathbf{j}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \ \underline{\mathbf{k}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

we have

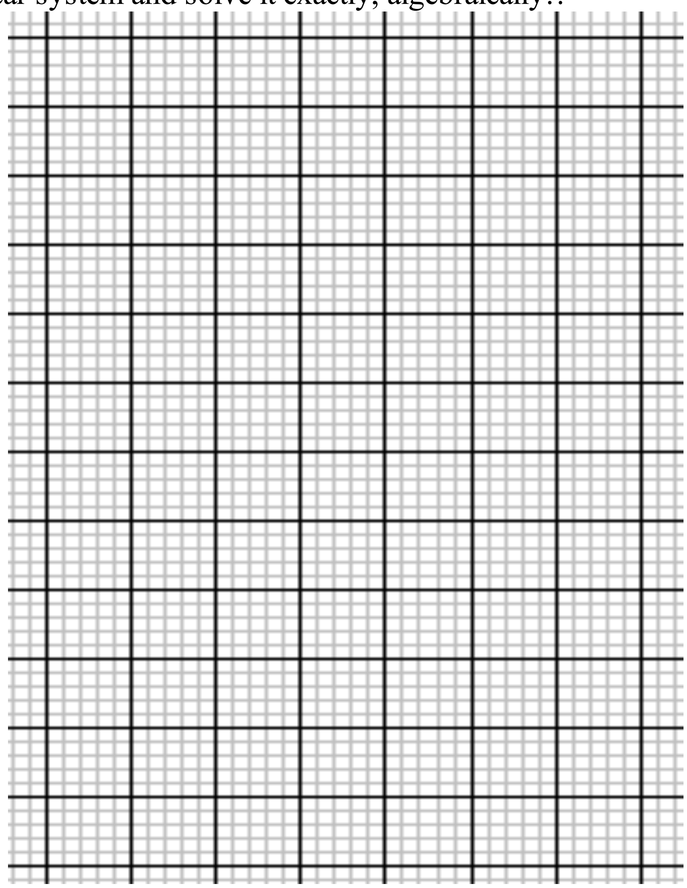
$$x\,\underline{\mathbf{i}} + y\,\underline{\mathbf{j}} + z\,\underline{\mathbf{k}} = x\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Exercise 4) Can you get to the point $(-2, 8) \in \mathbb{R}^2$, from the origin $(0, 0)$, by moving only in the $(\pm)$ directions of $\boldsymbol{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$? Algebraically, this means we want to solve the linear combination problem

$$x_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 8 \end{bmatrix}.$$

4a) Superimpose a grid related to the displacement vectors $\boldsymbol{u}$, $\boldsymbol{v}$ onto the graph paper below, and, recalling that vector addition yields net displacement, and scalar multiplication yields scaled displacement, try to approximately solve the linear combination problem above, geometrically.

4b) Rewrite the linear combination problem as a linear system and solve it exactly, algebraically!!



1c) Can you get to any point (x, y) in $\mathbb{R}^2$, starting at $(0, 0)$ and moving only in directions parallel to $\boldsymbol{u}$, $\boldsymbol{v}$? Argue geometrically and algebraically. How many ways are there to express $\begin{bmatrix} x \\ y \end{bmatrix}$ as a linear combination of $\boldsymbol{u}$ and $\boldsymbol{v}$?

Math 2250-004
Fri Jan 12
- 1.3 linear combinations and vector equations, continued.
- food for thought in second half of class

• try to finish 1.1 & 1.2 HW (try 1.3 if you're bold)

Announcements:

- MLK day is Monday
- Use Wed notes today (before f.f.t. lab). 1.3 due Tues.
- I'll stay late today (until 2:45
- I'll bring class notes for next week on Tuesday

Warm-up Exercise:
- Which of these augmented matrices is in reduced row echelon form?
- Do the associated systems of linear equations have
  - (i) no solutions (inconsistent)
  - (ii) a unique solution (i.e. exactly one sol'tn)
  - (iii) ∞'ly many solutions

unique soltn a) $\left[\begin{array}{ccc|c} 1 & 2 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 3 & 15 \end{array}\right]$ NO

$2 \neq 0$, $3 \neq 1$

no soltns, (inconsistent) last eqtn says $0=1$ b) $\left[\begin{array}{ccc|c} 1 & 0 & 17 & \cancel{3} \\ 0 & 1 & 6 & \cancel{2} \\ 0 & 0 & 0 & 1 \end{array}\right]$ needed to be zero NO pivots are O.K.

$\left[\begin{array}{ccc|c} 1 & 0 & 17 & 0 \\ 0 & 1 & 6 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$ is in rref

unique $y=2$ $x=5$ c) $\left[\begin{array}{cc|c} 0 & 1 & 2 \\ 1 & 0 & 5 \\ 0 & 1 & 2 \end{array}\right]$ NO 1st non-zero row entries don't move right as you go down rows

∞'ly many (1 free parameter because 3rd column is not pivot column d) $\left[\begin{array}{ccc|c} 1 & 0 & 17 & 3 \\ 0 & 1 & 16 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$ YES!

$$\begin{aligned} x \quad + 17z &= 3 \\ y + 16z &= 2 \end{aligned}$$

Soltns are
$x = 3 - 17z$
$y = 2 - 16z$
z free