Math 2270-004
Homework due March 7.

Recall that problems which are not underlined are good for seeing if you can work with the underlying concepts; only the underlined problems need to be handed in. The Wednesday quiz will be drawn from all of these concepts and from these or related problems.

Remember that you are allowed (or even encouraged) to use technology in order to compute the reduced row echelon form of any matrix with more than 12 entries.

4.4 Coordinate systems

1, **3**, **5**, **9**, **11**, **13**, **15**, **21**, **27**, **31**

4.*5: Dimension of vector spaces*

3, **7**, **9**, **11**, 13, **19**, **25**, **27**, **28** (hint: use 27). *In 27, 28, we say that a vector space is infinite dimensional if there is not basis consisting of a finite number of vectors.*

4.6: rank, and the four fundamental subspaces of a matrix.

1, **5**, **7**, **9**, **11**, **17**, 19, 21, **27.**

w8.1) In last week's homework you considered the matrix $A_{3 \times 6}$ given by

$$A := \begin{bmatrix} 1 & 2 & -1 & 2 & -1 \\ 2 & 4 & 0 & 6 & 2 \\ 1 & 2 & 2 & 5 & 5 \end{bmatrix}.$$

And found bases for *Col A* and *Nul A*. You used the reduced row echelon form of *A*, which is given by

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

w8.1a) Find a basis for *Row A*, the subspace of $\mathbb{R}^5$ spanned by the three rows of *A*. Hint: There is a very good basis available in one of the matrices above. What is the dimension of this subspace of $\mathbb{R}^5$?

w8.1b) Explain why the dimension of *Row B* always equals the dimension of *Col B*, for any matrix. Hint: Look at the what the reduced row echelon form of *B* tells you about these dimensions. The common value is called the *rank* of the matrix *B*.

w8.1.c) Last week you understood the "rank plus nullity" theorem, namely that

$$dim(Nul\,B) + dim(Col\,B) = n$$

Notice that because the transpose operation interchanges rows and columns, *Row B* is actually $Col\left(B^T\right)$.

So applying the rank plus nullity theorem to the transpose matrix, we see that

$$dim\left(Nul\ B^T\right) + dim\left(Col\ B^T\right) = m.$$
$$dim\left(Nul\ B^T\right) + dim\left(Row\ B\right) = m.$$

<u>w8.1d</u>) In our example, A is a 3×5 matrix and A^T is 5×3. Find a basis for $Nul\left(A^T\right)$ and verify that in your example

$$dim\left(Nul\ A^T\right) + dim\left(Row\ A\right) = 3.$$

(Hint: when you column reduced A to find a very good basis for $Col\ A$, you were doing the same computations as you need to do to row reduce A^T.

<u>w8.2)</u> Use the two dimension equalities from <u>w8.1c</u> above to explain why it is true that for any $m \times n$ matrix B, that the matrix equation

$$B\,\underline{\mathbf{x}} = \underline{\boldsymbol{b}}$$

is consistent for all $\underline{\boldsymbol{b}} \in \mathbb{R}^m$ if and only if the homogeneous equation

$$B^T\underline{\mathbf{x}} = \underline{\mathbf{0}}$$

has only the trivial solution $\underline{\mathbf{x}} = \underline{\mathbf{0}}$. (This is the surprising generalization of a fact we knew about square matrices, to general rectangular ones, because in the case that B is $n \times n$ these are two of the equivalent statements on our the very long list of things equivalent to B having an inverse matrix.)