Wed Feb 7

- 2.2-2.3 Matrix inverses: the product of elementary matrices approach to matrix inverses

Announcements:
- on exam, you'll need to know basic definitions (closed book, closed note)
- any hw Q's
- today: massive time to review some of material
- tomorrow: practice exam (review sheet?) posted later today on CANVAS, go over in review session JWB 335 12:55-2:15

Warm-up Exercise: Definitions: e.g. 'til 12:57

a) $\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n\}$ are linearly independent means

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n = \vec{0} \implies c_1 = c_2 = \ldots = c_n = 0$$

b) $\text{span}\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n\} = \{x_1\vec{v}_1 + x_2\vec{v}_2 + \ldots + x_n\vec{v}_n$, such that each $x_j \in \mathbb{R}$, $j = 1, 2, \ldots n\}$

c) $T: \mathbb{R}^n \to \mathbb{R}^m$ linear transformation

$$T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v}) \qquad \forall\, \vec{u}, \vec{v} \in \mathbb{R}^n$$

$$T(c\vec{u}) = cT(\vec{u}) \qquad \forall\, \vec{u} \in \mathbb{R}^n,\ c \in \mathbb{R}$$

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- we showed that matrix transformations are linear: $T(\vec{x}) = A\vec{x}$
  $A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y}$
  $A(c\vec{x}) = cA\vec{x}$
- we showed that these linear transformations are matrix trans.

the matrix for T is given by

$$\left[\begin{array}{c:c:c:c} T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_n) \end{array}\right]$$

<u>Exercise 1)</u> Show that if A, B, C are invertible matrices, then

$$(A\,B)^{-1} = B^{-1}\,A^{-1}.$$
$$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$$

<u>Theorem</u> The product of $n \times n$ invertible matrices is invertible, and the inverse of the product is the product of their inverses in reverse order.

Saying the same thing in lots of different ways (important because it ties a lot of our Chapter 1-2 ideas together): Can you explain why these are all equivalent?

<u>The invertible matrix theorem</u> (page 114)

Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

a) A is an invertible matrix.

b) The reduced row echelon form of A is the $n \times n$ identity matrix.

c) A has n pivot positions

$a \Rightarrow b$ if A^{-1} exists, then the solution $\vec{x}$ to $A\vec{x} = \vec{b}$ is $\vec{x} = A^{-1}\vec{b}$

so the solution $\vec{x}$ is unique

- since each such eqtn has a solution there's a pivot in each row

n pivots same as $\text{rref}(A) = I$

(• or, solutions are unique so no free variables, so each column has a pivot so n pivots)

$b \Leftrightarrow c$

$b \Rightarrow a$ via our algorithm. (augment A with I, reduce)

d) The equation $A\,\underline{\boldsymbol{x}} = \underline{\mathbf{0}}$ has only the trivial solution $\underline{\boldsymbol{x}} = \underline{\mathbf{0}}$.

write

$$A = [\vec{a}_1, \vec{a}_2, \ldots \vec{a}_n]$$

e) The columns of A form a linearly independent set.

f) The linear transformation $T(\underline{\boldsymbol{x}}) := A\,\underline{\boldsymbol{x}}$ is one-one.

$d \Rightarrow e.$ want to show that

(1) $c_1\vec{a}_1 + c_2\vec{a}_2 + \cdots + c_n\vec{a}_n = \vec{0}$ then $c_1 = c_2 = \cdots = c_n = 0$

$$\parallel$$

(2) $A\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \vec{0}$

if d is true $\vec{c} = \vec{0}$, so $c_1 = c_2 = \cdots = c_n = 0$

$e \Rightarrow d$: just read previous paragraph backwards : If e is true then (1) is true, so the only sol'n to (2) is $\vec{c} = \vec{0}$.

$d \Rightarrow f$ $T(\vec{x})$ one-one means the problem $T(\vec{x}) = \vec{b}$ always has unique solutions.

$f \Rightarrow d.$ if solns to $A\vec{x} = \vec{b}$ are unique, then the only solution to $A\vec{x} = \vec{0}$ is $\vec{x} = \vec{0}$!

if $A\vec{x} = \vec{b}$ & $A\vec{y} = \vec{b}$ ~~and~~ is $\vec{x} = \vec{y}$?

$$A\vec{x} - A\vec{y} = \vec{0}$$
$$A(\vec{x} - \vec{y}) = \vec{0}$$

if (d) is true, $\vec{x} - \vec{y} = \vec{0}$, $\vec{x} = \vec{y}$ ∎

connect abc, def

$a \Rightarrow d$: if A^{-1} exists the solution to $A\vec{x} = \vec{0}$ $\vec{x} = A^{-1}\vec{0} = \vec{0}$

$d \Rightarrow b$: no free parameter $\Rightarrow$ rref$(A) = I$

g) The equation $A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}$ has at least one solution for each $\underline{\boldsymbol{b}} \in \mathbb{R}^n$.

h) The columns of A span $\mathbb{R}^n$.

i) The linear transformation $T(\underline{\boldsymbol{x}}) := A\,\underline{\boldsymbol{x}}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.

g), h), i) are saying exactly the same thing, but in different contexts.

h) Write $A = [\vec{a}_1\ \vec{a}_2 \cdots \vec{a}_n]$ in terms of its columns.

$$\mathrm{span}\{\vec{a}_1, \vec{a}_2, \ldots \vec{a}_n\} = \left\{ x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n \text{ such that } \begin{matrix} x_j \in \mathbb{R} \\ 1 \le j \le n \end{matrix} \right\}$$

$$= \{ A\vec{x} \text{ such that } \vec{x} \in \mathbb{R}^n \}$$

So, saying each $\vec{b} \in \mathbb{R}^n$ is in the span of the columns of A is saying the equation $A\vec{x} = \vec{b}$ always has at least one solution

So, $g \Leftrightarrow h$.

i) a function $f: X \to Y$ is <u>onto</u> means the equation $f(x) = b$ always has at least one solution x, for each $b \in Y$.

So asking whether $T: \mathbb{R}^n \to \mathbb{R}^m$ is onto, is asking whether the equation $A\vec{x} = \vec{b}$ always has at least one solution

So $g \Leftrightarrow i$

connect g, h, i to a, b, c:

$b \Rightarrow g$: iff $\mathrm{rref}(A) = I$ then the eqtn $A\vec{x} = \vec{b}$ always has a solution, because $\mathrm{rref}(A)$ has a pivot in each row.

$g \Rightarrow b$: If $A\vec{x} = \vec{b}$ always has at least one sol'n, $\mathrm{rref}(A)$ has a pivot in each row, so since A is a square matrix, $\mathrm{rref}(A) = I$

j) There is an $n \times n$ matrix C such that $CA = I$.

k) There is an $n \times n$ matrix D such that A D $= I$.

l) A^T is an invertible matrix.

a) $\Rightarrow$ j, k : if A^{-1} exists, then let $C = A^{-1}$ for j)
let $D = A^{-1}$ for k)

a $\Rightarrow$ l , recall that $(AB)^T = B^T A^T$
so, if A has an inverse, A^{-1} is the matrix B for which

$$AB = I \quad \& \quad BA = I$$

Therefore $(AB)^T = I^T$; $(BA)^T = I^T$

$B^T A^T = I$; $A^T B^T = I$

So A^T has an inverse, given by $(A^T)^{-1} = (A^{-1})^T$

j $\Rightarrow$ a) :

if $CA = I$
then the only solution $\vec{x}$ to $A\vec{x} = \vec{0}$ is $\vec{x} = 0$, since

$$A\vec{x} = \vec{0}$$
$$\Rightarrow CA\vec{x} = C\vec{0} = \vec{0}$$
$$\Rightarrow I\vec{x} = \vec{0} \Rightarrow \vec{x} = \vec{0}.$$

So, j $\Rightarrow$ d $\Rightarrow$ a

k $\Rightarrow$ a)

If $AD = I$
then $AD\vec{b} = I\vec{b} = \vec{b}$ for any $\vec{b} \in \mathbb{R}^n$.
so the eqtn $A\vec{x} = \vec{b}$ has a solution $\vec{x} = D\vec{b}$.

So k $\Rightarrow$ g $\Rightarrow$ a

l $\Rightarrow$ a) : If $(A^T)^{-1}$ exists, then apply a $\Rightarrow$ l above to the matrix A^T to see that $(A^T)^T = A$ has an inverse

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