Wed Feb 28

- 4.5 dimension of a vector space, and related facts about span and linear independence.

Announcements:
- Tuesday's notes today.
- Quiz

'til 10:57

Warm-up Exercise:

$$\begin{array}{cccc} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 & \vec{a}_4 \end{array} \quad \text{col } A = \text{span}\{\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4\}$$

$$A = \begin{bmatrix} 1 & 3 & 0 & 4 \\ 2 & -2 & -8 & 0 \\ -1 & 2 & 5 & 1 \\ 0 & 4 & 4 & 4 \end{bmatrix} \qquad \text{rref}(A) = \begin{bmatrix} 1 & 0 & -3 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

what is the dimension of Col A? $= 2$ (# of vectors in a basis)

a basis for Col A? $\{\vec{a}_1, \vec{a}_2\} = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 2 \\ 4 \end{bmatrix} \right\}$

what is dimension of Nul A?

could you find a basis of Nul A?

$\dim \text{Nul } A = 2 =$ # of free variables when we solve $A\vec{x} = \vec{0}$

$=$ # columns of rref(A) without pivots.

From Tuesday's notes about bases yielding coordinate systems; covered Wednesday

<u>Theorem</u> Let V be a vector space, and let $\{\underline{\boldsymbol{b}}_1, \underline{\boldsymbol{b}}_2, \ldots \underline{\boldsymbol{b}}_p\}$ be a basis for V. Then for each $\underline{\boldsymbol{v}} \in V$ there is a unique set of scalars $c_1, c_2, \ldots c_p$ so that

$$\underline{\boldsymbol{v}} = c_1\underline{\boldsymbol{b}}_1 + c_2\underline{\boldsymbol{b}}_2 + \ldots + c_p\underline{\boldsymbol{b}}_p. \qquad E_1$$

<u>proof:</u> Suppose we could also write $\vec{v} = d_1\vec{b}_1 + d_2\vec{b}_2 + \cdots + d_p\vec{b}_p \qquad E_2$

subtract E_2 from E_1 (and use vector space axioms)

$$\vec{0} = c_1\vec{b}_1 + c_2\vec{b}_2 + \cdots + c_p\vec{b}_p - (d_1\vec{b}_1 + d_2\vec{b}_2 + \cdots + d_p\vec{b}_p)$$

commutativity, associativity

$$= (c_1\vec{b}_1 - d_1\vec{b}_1) + (c_2\vec{b}_2 - d_2\vec{b}_2) + \cdots + (c_p\vec{b}_p - d_p\vec{b}_p)$$

$$\vec{0} = (c_1 - d_1)\vec{b}_1 + (c_2 - d_2)\vec{b}_2 + \cdots + (c_p - d_p)\vec{b}_p$$

because $\{\vec{b}_1, \vec{b}_2, \ldots \vec{b}_p\}$ in lin. indep.

know $c_1 - d_1 = 0$

$c_2 - d_2 = 0$

$\vdots$

$c_p - d_p = 0$

i.e. each $c_j = d_j$.

$\blacksquare$

From Tuesday's notes, covered Wed.

Definition (Each basis gives us a coordinate system). Let V be a vector space, and let $\beta = \{\underline{\boldsymbol{b}}_1, \underline{\boldsymbol{b}}_2, \ldots \underline{\boldsymbol{b}}_p\}$ be a basis for V. For each $\underline{\boldsymbol{v}} \in V$ we say that *the coordinates of* $\underline{\boldsymbol{v}}$ *with respect to* β are $c_1, c_2, \ldots c_p$ if

$$\underline{\boldsymbol{v}} = c_1\underline{\boldsymbol{b}}_1 + c_2\underline{\boldsymbol{b}}_2 + \ldots + c_p\underline{\boldsymbol{b}}_p.$$

And, we write the vector of the coordinates of $\underline{\boldsymbol{v}}$ with respect to β as:

$$[\underline{\boldsymbol{v}}]_\beta = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_p \end{bmatrix} \in \mathbb{R}^p .$$

coord. vector of $\vec{v}$ with respect to the basis β.

Example: For the vector space

$$P_3 = \left\{p(t) = a_0 + a_1\, t + a_2\, t^2 + a_3\, t^3 \text{ such that } a_0, a_1, a_2, a_3 \in \mathbb{R}\right\}$$

we've checked that

$$\beta = \{\, 1, t, t^2, t^3 \,\}$$

is a basis. So the coordinate vector of

$$p(t) = 3 - 4\,t^2 + t^3$$

$0t$

with respect to β is

$$[p]_\beta = \begin{bmatrix} 3 \\ 0 \\ -4 \\ 1 \end{bmatrix}$$

From Tuesday's notes, covered Wednesday

And, if $q \in P_3$, with

$$[q]_\beta = \begin{bmatrix} -2 \\ 1 \\ 7 \\ 0 \end{bmatrix}$$

then

$$q(t) = -2 + t + 7t^2.$$

It turns out that we can understand pretty much any vector space question about P_3 by interpreting the question in terms of the coordinates with respect to β, which lets us work in $\mathbb{R}^4$ in lieu of P_3. That's what coordinates with respect to a basis are good for, when you're working with a non-standard vector space.

From Tuesday's notes, covered Wed.

Exercise 1) Let

$$\beta = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\} = \{\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}}\}$$

be a non-standard basis of $\mathbb{R}^2$.

1a) Suppose $\underline{\boldsymbol{x}}$ is a vector in $\mathbb{R}^2$, and

$$[\underline{\boldsymbol{x}}]_\beta = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Find the standard coordinates for $\underline{\boldsymbol{x}}$, i.e. its coordinates with respect to the standard basis $E = \{\underline{\boldsymbol{e}}_1, \underline{\boldsymbol{e}}_2\}$.

$$\Rightarrow \vec{x} = 2\begin{bmatrix} 1 \\ -1 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} = [\vec{x}]_E$$

$$\vec{x} = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$[\vec{x}]_E = \underbrace{\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}}_{P_{E \leftarrow \beta}} [\vec{x}]_\beta$$

transition matrix from β coords to standard (E) coords

$$[\vec{x}]_\beta = \overbrace{\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}^{-1}}^{P_{\beta \leftarrow E}} [\vec{x}]_E$$

1b) Find the β - coordinates for the vector $\underline{\boldsymbol{b}} = \begin{bmatrix} -2 \\ 8 \end{bmatrix}$. (The math may seem familiar.)

1c) Interpret your work in 1ab geometrically, in terms of the coordinate system generated by β.

1b). $\vec{b} = \begin{bmatrix} -2 \\ 8 \end{bmatrix}$

$[b]_\beta$: $c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 8 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 8 \end{bmatrix}$$

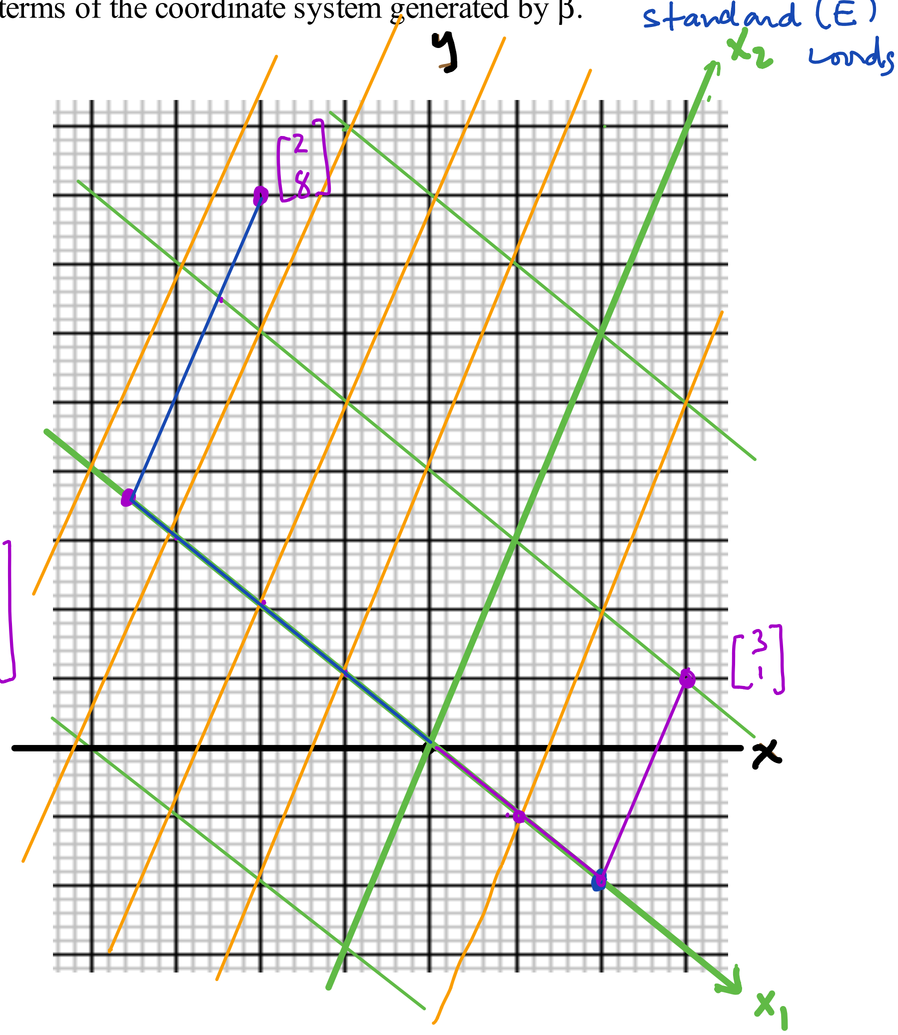
$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ 8 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} -14 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} -3.5 \\ 1.5 \end{bmatrix}$$

So

$$[\vec{b}]_\beta = \begin{bmatrix} -3.5 \\ 1.5 \end{bmatrix}$$



From Tuesday's notes, covered Wed.

Theorem Let V, W be vector spaces, and $T : V \to W$ a linear transformation. If T is $1 - 1$ and onto, then the inverse function T^{-1} is also a linear transformation, $T^{-1} : W \to V$. In this case, we call T an *isomorphism*.

proof: We have to check that for all $\underline{\boldsymbol{u}}, \underline{\boldsymbol{w}} \in W$ and all $c \in \mathbb{R}$,

check

$$T^{-1}(\underline{\boldsymbol{u}} + \underline{\boldsymbol{w}}) = T^{-1}(\underline{\boldsymbol{u}}) + T^{-1}(\underline{\boldsymbol{w}})$$

$$T^{-1}(c\,\underline{\boldsymbol{u}}) = c\,T^{-1}(\underline{\boldsymbol{u}}).$$

Since T is 1-1

suffices to check that T of LHS's = T RHS's

T linear

T of RHS $T\left(T^{-1}(\vec{u}) + T^{-1}(\vec{w})\right) \stackrel{\downarrow}{=} T(T^{-1}(\vec{u})) + T(T^{-1}(\vec{w}))$

$= \vec{u} + \vec{w}$

T of LHS. : $T(T^{-1}(\vec{u}+\vec{w}))$

$= \vec{u} + \vec{w}$

T is linear

$T(T^{-1}(c\vec{u})) = c\vec{u}$ $\qquad$ $T(cT^{-1}(\vec{u})) \stackrel{\downarrow}{=} c\,TT^{-1}(\vec{u})$

$= c\vec{u}$

Theorem Let V be a vector space, with basis $\beta = \{\underline{\boldsymbol{b}}_1, \underline{\boldsymbol{b}}_2, \ldots \underline{\boldsymbol{b}}_n\}$. Then the coordinate transform $T : V \to \mathbb{R}^n$ defined by

$$T(\underline{\boldsymbol{v}}) = [\underline{\boldsymbol{v}}]_\beta$$

is linear, and it is an isomorphism.