Math 2270-004 Week 8 notes
We will not necessarily finish the material from a given day's notes on that day. We may also add or subtract some material as the week progresses, but these notes represent an in-depth outline of what we plan to cover. These notes cover material in 4.2-4.6.

Mon Feb 26
- 4.2 - 4.3 bases for vector spaces and subspaces; *Nul A* and *Col A*; generalization to linear transformations.

Announcements: I'll post fft soltns tonight

'til for $\mathbb{R}^n$ our favorite basis is $\{\vec{e}_1, \vec{e}_2, \ldots \vec{e}_n\}$. But for a subspace of $\mathbb{R}^n$, what's a "best" basis? $\boxed{\text{no row operations required}}$

Warm-up Exercise:

a) check that $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ is linearly independent. Which entries do you focus on?

b) express $\begin{bmatrix} 8 \\ 2 \\ -2 \\ 3 \\ -1 \end{bmatrix}$ as a linear combination of the vectors above. Which entries do you focus on?

a) $c_1 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{matrix} \\ \leftarrow 2^{nd} \\ \\ \leftarrow 4^{th} \\ \leftarrow 5^{th} \end{matrix}$

2^{nd}, 4^{th}, 5^{th} entries: $c_1 = 0$, $c_2 = 0$, $c_3 = 0$

almost as good as standard basis vectors.

b) $\underset{2}{c_1} \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \underset{3}{c_2} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \\ -2 \\ 3 \\ -1 \end{bmatrix}$

2^{nd} entry $c_1 = 2$
4^{th} entry $c_2 = 3$
5^{th} $c_3 = -1$

$2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \\ -2 \\ 3 \\ -1 \end{bmatrix}$ ✓ ✓

Monday Review!

We've been discussing *vector spaces*, which are a generalization of $\mathbb{R}^n$: Namely, a *vector space* is a nonempty set V of objects, called *vectors*, on which are defined two operations, called *addition* and *scalar multiplication,* so that ten natural axioms about vector addition and scalar multiplication hold (along with three additional useful consequences that we often use, and that you thought about on your food for thought).

Last week we discovered that certain subsets of vector spaces are also vector spaces (with the same addition and scalar multiplication as in the larger space) - namely *subspaces* of a vector space V: these are subsets H of V that satisfy

sub vector spaces.

a) The zero vector of V is in H
b) H is closed under vector addition, i.e. for each $\underline{\boldsymbol{u}} \in H, \underline{\boldsymbol{v}} \in H$ then $\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}} \in H$.
c) H is closed under scalar multiplication, i.e for each $\underline{\boldsymbol{u}} \in H$, $c \in \mathbb{R}$, then also $c\,\underline{\boldsymbol{u}} \in H$.

We defined *linear dependence* and *linear independence* for sets of vectors $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_p\}$ in a vector space V.

A *basis* for a vector space V is a set of vectors $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_p\}$ that *span* V and that is also *linearly* independent.

The *dimension* of a vector space V is the number of vectors in any basis for V. (We'll show why every basis for a fixed vector space V - no matter how weird V may seem - has the same number of vectors, later this week.)

We showed that one way subspaces arise is as $H = span\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ ... \ \underline{\boldsymbol{v}}_p\}$ for sets of vectors $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ ... \ \underline{\boldsymbol{v}}_p\}$ in a vector space V. This is an explicit way to describe H because you are saying exactly which vectors are in it. If the vectors in the spanning set $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ ... \ \underline{\boldsymbol{v}}_p\}$ are not already independent, we illustrated how to remove extraneous dependent vectors without shrinking the span, until we were left with a basis for the subspace H. (We'll return to this today .... it was an example with $H = Col\ A$)

We discovered that the only subsets of $\mathbb{R}^3$ that succeed at being subspaces of $\mathbb{R}^3$ are

- $\{\ \underline{\mathbf{0}}\ \}$ 0-dim'l, by def.
- $span\{\underline{\boldsymbol{u}}\}$ for some $\underline{\boldsymbol{u}} \neq \underline{\mathbf{0}}$ (a line thru the origin) 1 - dimensional subspaces
- $span\{\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}}\}$ for some $\{\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}}\}$ linearly independent 2 - dimensional subspaces
- $span\{\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}}, \underline{\boldsymbol{w}}\} = \mathbb{R}^3$ for $\{\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}}, \underline{\boldsymbol{w}}\}$ linearly independent 3 - dimensional (sub)space.

Note
$\mathbb{R}^2 = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : x_1, x_2 \in \mathbb{R} \right\}$.
a 2-dim'l plane in $\mathbb{R}^3$, thru $\vec{0}$, i.e. $span\{\vec{u}, \vec{v}\}$ $\vec{u}, \vec{v}$ ind. is not $\mathbb{R}^2$

- We realized that what happens in $\mathbb{R}^3$ with respect to subspaces, generalizes to $\mathbb{R}^n$.

Towards the end of class on Friday we realized that for an $m \times n$ matrix A,

$$Nul\ A := \{\underline{\boldsymbol{x}} \in \mathbb{R}^n \text{ for which } A\,\underline{\boldsymbol{x}} = \underline{\mathbf{0}}\}$$

is a subspace. This is an implicit way to specify a subspace, because you're prescribing equations which the elements $\underline{\boldsymbol{x}}$ musts satisfy, but not explicitly saying what the elements are.

Picking up where we left off ....

Monday Feb. 26
finish from Friday

Nul A

Exercise 1a) For the same matrix A as in Exercise 2 from Wednesday's notes, express the vectors in $Nul(A)$ explicitly, using the methods of Chapters 1-2. Notice these are vectors in the domain of the associated linear transformation $T: \mathbb{R}^5 \to \mathbb{R}^3$ given by $T(\underline{\boldsymbol{x}}) = A\,\underline{\boldsymbol{x}}$, so are a subspace of $\mathbb{R}^5$.

$A\vec{x} = \vec{0}$

$$A = \left[\begin{array}{ccccc|c} 1 & -2 & 0 & -1 & 1 & 0 \\ -2 & 4 & 0 & 2 & -2 & 0 \\ 3 & -6 & 4 & 1 & 7 & 0 \end{array}\right] \text{ reduces to } \left[\begin{array}{ccccc|c} 1 & -2 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\boxed{\{\vec{x} : A\vec{x} = \vec{0}\} \quad \text{implicit description}}$

$x_1 = 2t_2 + t_4 - t_5$
$x_2 = t_2$ free
$x_3 = -t_4 - t_5$
$x_4 = t_4$ free
$x_5 = t_5$ free

explicit description
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = t_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t_4 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t_5 \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

1b) Exhibit a basis for $Nul(A)$.

$$\text{Nul } A = \text{span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

free params came from non-pivot columns, e.g.
$x_2 = t_2$ free.
$x_4 = t_4$ free
$x_5 = t_5$ free

as consequence,
2nd entry of dep. eqtn says $c_1 = 0$
4th entry says $c_2 = 0$
5th entry says $c_3 = 0$

where we ended. [actually a basis]
needed to check independence. that was warmup prob.

if
$$c_1 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

OR

$$t_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t_4 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t_5 \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Monday Feb 26
Review

from Wed. notes, covered Friday.

subspace as a span

4.2 Null spaces, column spaces, and linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$.

Definition Let A be an $m \times n$ matrix, expressed in column form as $A = [\underline{\boldsymbol{a}}_1\ \underline{\boldsymbol{a}}_2\ \underline{\boldsymbol{a}}_3\ \ldots\ \underline{\boldsymbol{a}}_n]$ The *column space* of A, written as $Col\,A$, is the span of the columns:

$$Col\,A = span\{\underline{\boldsymbol{a}}_1\ \underline{\boldsymbol{a}}_2\ \underline{\boldsymbol{a}}_3\ \ldots\ \underline{\boldsymbol{a}}_n\}\,.$$

Equivalently, since

$$A\,\underline{\boldsymbol{x}} = x_1\,\underline{\boldsymbol{a}}_1 + x_2\,\underline{\boldsymbol{a}}_2 + \ldots + x_n\,\underline{\boldsymbol{a}}_n$$

we see that $Col\,A$ is also the range of the linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ given by $T(\underline{\boldsymbol{x}}) = A\,\underline{\boldsymbol{x}}$, i.e

$$Col\,A = \{\underline{\boldsymbol{b}} \in \mathbb{R}^m \text{ such that } \underline{\boldsymbol{b}} = A\,\underline{\boldsymbol{x}} \text{ for some } \underline{\boldsymbol{x}} \in \mathbb{R}^n\}.$$

Theorem By the "spans are subspaces" theorem, $Col(A)$ is always a subspace of $\mathbb{R}^m$.

Exercise 2a) Consider

$$A = \begin{bmatrix} 1 & -2 & 0 & -1 & 1 \\ -2 & 4 & 0 & 2 & -2 \\ 3 & -6 & 4 & 1 & 7 \end{bmatrix}.$$

By the Theorem, $col(A)$ is a subspace of $\mathbb{R}^3$. Which is it: $\{\underline{\boldsymbol{0}}\}$, a line thru the origin, a plane thru the origin, or all of $\mathbb{R}^3$. Hint:

$$\begin{array}{ccccc} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 & \vec{a}_4 & \vec{a}_5 \end{array} \qquad\qquad \begin{array}{ccccc} \vec{r}_1 & \vec{r}_2 & \vec{r}_3 & \vec{r}_4 & \vec{r}_5 \end{array}$$

$$\begin{bmatrix} 1 & -2 & 0 & -1 & 1 \\ -2 & 4 & 0 & 2 & -2 \\ 3 & -6 & 4 & 1 & 7 \end{bmatrix} \text{ reduces to } \begin{bmatrix} 1 & -2 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

$\vec{a}_1 \neq \vec{0}$
$\vec{a}_2 = -2\vec{a}_1$
$\vec{a}_3$ ind of $\vec{a}_1$
$\vec{a}_4 = -\vec{a}_1 + \vec{a}_3$
$\vec{a}_5 = \vec{a}_1 + \vec{a}_3$

$\Longleftarrow$

$\vec{r}_1 \neq \vec{0}$
$\vec{r}_2 = -2\vec{r}_1$
$\vec{r}_3$ ind. of $\vec{r}_1$
$\vec{r}_4 = -\vec{r}_1 + \vec{r}_3$
$\vec{r}_5 = \vec{r}_1 + \vec{r}_3$

2b) Is there a more efficient way to express $Col\,A$ as a span that doesn't require all five column vectors?

YES.

$$c_1\vec{a}_1 + c_2\vec{a}_2 + c_3\vec{a}_3 + c_4\vec{a}_4 + c_5\vec{a}_5$$
$$= c_1\vec{a}_1 + c_2(-2\vec{a}_1) + c_3\vec{a}_3 + c_4(-\vec{a}_1 + \vec{a}_3) + c_5(\vec{a}_1 + \vec{a}_3)$$
$$= d_1\vec{a}_1 + d_3\vec{a}_3$$

$$Col\,A = span\{\vec{a}_1, \vec{a}_2, \ldots \vec{a}_5\} = span\{\vec{a}_1, \vec{a}_3\}$$

so, col A was just a plane thru origin

Not all bases are created equal!

<u>Theorem</u>: Let $span\{\underline{v}_1, \underline{v}_2, \ldots \underline{v}_p\} = H$ be a subspace. The following *elementary operations* do not effect the span of the resulting ordered set:

(i) swap two of the vectors in the set, i.e. replace $\underline{v}_j$ with $\underline{v}_k$, and replace $\underline{v}_k$ with $\underline{v}_j$.

$$span\{\vec{v}_1, .. \vec{v}_j, .. \vec{v}_k, .. \vec{v}_p\} = span\{\vec{v}_1, .. \vec{v}_k, .. \vec{v}_j, .. \vec{v}_p\}$$

(ii) replace $\underline{v}_j$ with $c\,\underline{v}_j$, for $c \neq 0$.

$$span\{\vec{v}_1, .. \vec{v}_j, .. \vec{v}_p\} = span\{\vec{v}_1, .. c\vec{v}_j, .. \vec{v}_p\}$$

$$c_1\vec{v}_1 + c_2\vec{v}_2 + .. + c_p\vec{v}_p = c_1\vec{v}_1 + c_2\vec{v}_2 + .. + \frac{c_j}{c} c\vec{v}_j + \ldots + c_p\vec{v}_p$$

$$.. + c_j\vec{v}_j$$

(iii) for $j \neq k$, replace $\underline{v}_k$ with $\underline{v}_k + c\,\underline{v}_j$.

$$span\{\vec{v}_1, .. \vec{v}_j .. \vec{v}_k .. \vec{v}_p\} = span\{\vec{v}_1, .. \vec{v}_j .. (\vec{v}_k + c\vec{v}_j) .. \vec{v}_p\}$$

$$c_1\vec{v}_1 + c_2\vec{v}_2 + .. + c_k\vec{v}_k + .. + c_p\vec{v}_p = c_1\vec{v}_1 + \ldots (c_j - c c_k)\vec{v}_j + \ldots \quad c_k(\vec{v}_k + c\vec{v}_j) + \ldots + c_p\vec{v}_p$$

$$+ .. c_j\vec{v}_j ..$$

<u>Exercise 1)</u> Use the "change of spanning set" theorem above, to find a better basis for *Col A* then the one we came up with by culling dependent vectors, on Friday. Hint: Use elementary column operations to compute the reduced column echelon form of *A*. Illustrate why this new basis is a better basis for *Col A* by seeing how easy it is to express any one of the original column vectors in terms of this improved basis.

In this example, $A = [\underline{a}_1\ \underline{a}_2\ \underline{a}_3\ \underline{a}_4\ \underline{a}_5]$ and $Col\,A = span\{\underline{a}_1, \underline{a}_2, \underline{a}_3, \underline{a}_4, \underline{a}_5\} = span\{\vec{a}_1, \vec{a}_3\}$.

$$A = \begin{bmatrix} 1 & -2 & 0 & -1 & 1 \\ -2 & 4 & 0 & 2 & -2 \\ 3 & -6 & 4 & 1 & 7 \end{bmatrix}.$$

As we just reviewed, on Friday we realized that a pretty good basis for *Col A* is $\{\underline{a}_1, \underline{a}_3\}$:

$$\begin{bmatrix} 1 & -2 & 0 & -1 & 1 \\ -2 & 4 & 0 & 2 & -2 \\ 3 & -6 & 4 & 1 & 7 \end{bmatrix} \quad \text{row reduces to} \quad \begin{bmatrix} 1 & -2 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Now column reduce A to get a basis for *Col A* that's as good as you could hope for....and show this by expressing each of the original columns in terms of this basis.

$$A = \begin{array}{ccccc} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 & \vec{a}_4 & \vec{a}_5 \\ \hline \end{array}$$

$$A = \begin{bmatrix} 1 & -2 & 0 & -1 & 1 \\ -2 & 4 & 0 & 2 & -2 \\ 3 & -6 & 4 & 1 & 7 \end{bmatrix}$$

better Col A basis

rcef!

$2\vec{a}_1 + \vec{a}_2 \to \vec{a}_2$
$\vec{a}_1 + \vec{a}_4 \to \vec{a}_4$
$-\vec{a}_1 + \vec{a}_5 \to \vec{a}_5$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 \\ 3 & 0 & 4 & 4 & 4 \end{bmatrix}$$

$a_2 \to a_5$
$a_5 \to a_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 \\ 3 & 4 & 4 & 4 & 0 \end{bmatrix}$$

$\frac{1}{4}a_2 \to a_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 \\ 3 & 1 & 4 & 4 & 0 \end{bmatrix}$$

$-4\vec{a}_2 + \vec{a}_3 \to \vec{a}_3$
$-4\vec{a}_2 + \vec{a}_4 \to a_4$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$-3\vec{a}_2 + \vec{a}_1 \to \vec{a}_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\vec{a}_5 = 1\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + 7\begin{bmatrix} 1 \\ -2 \\ 7 \end{bmatrix}$$

$$= \left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

best basis for Col A