Math 2270-004 Week 8 notes

We will not necessarily finish the material from a given day's notes on that day. We may also add or subtract some material as the week progresses, but these notes represent an in-depth outline of what we plan to cover. These notes cover material in 4.2-4.6.

Mon Feb 26

• 4.2 - 4.3 bases for vector spaces and subspaces; *Nul A* and *Col A*; generalization to linear transformations.

Announcements: I'll post fft sollins tonight

2nd entry
$$c_1 = 2$$

4th entry $c_2 = 3$
5th $c_3 = -1$
2 $\binom{2}{1} + 3 \begin{pmatrix} 1\\0\\-1\\0\\0\\0 \end{pmatrix} = \binom{-1}{0} = \binom{8}{2}$
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Monday Review!

We've been discussing <u>vector spaces</u>, which are a generalization of \mathbb{R}^n : Namely, a vector space is a nonempty set *V* of objects, called vectors, on which are defined two operations, called *addition* and *scalar multiplication*, so that ten natural axioms about vector addition and scalar multiplication hold (along with three additional useful consequences that we often use, and that you thought about on your food for thought).

Last week we discovered that certain subsets of vector spaces are also vector spaces (with the same addition and scalar multiplication as in the larger space) - namely <u>subspaces</u> of a vector space V: these are subsets H of V that satisfy She vector space Spaces.

- a) The zero vector of V is in H
- b) *H* is closed under vector addition, i.e. for each $\underline{u} \in H$, $\underline{v} \in H$ then $\underline{u} + \underline{v} \in H$.
- c) *H* is closed under scalar multiplication, i.e for each $\underline{u} \in H$, $c \in \mathbb{R}$, then also $c \underline{u} \in H$.

We defined <u>linear dependence</u> and <u>linear independence</u> for sets of vectors $\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_p\}$ in a vector space V.

A *basis* for a vector space V is a set of vectors $\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_p\}$ that *span V* and that is also *linearly* independent.

The *dimension* of a vector space V is the number of vectors in any basis for V. (We'll show why every basis for a fixed vector space V- no matter how weird V may seem - has the same number of vectors, later this week.)

We showed that one way subspaces arise is as $H = span \{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_p\}$ for sets of vectors $\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_p\}$ in a vector space *V*. This is an explicit way to describe *H* because you are saying exactly which vectors are in it. If the vectors in the spanning set $\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_p\}$ are not already independent, we illustrated how to remove extraneous dependent vectors without shrinking the span, until we were left with a basis for the subspace *H*. (We'll return to this today ..., it was an example with H = Col A)

O-dimil, by def.

1 - dimensional subspaces

We discovered that the only subsets of \mathbb{R}^3 that succeed at being subspaces of \mathbb{R}^3 are

• { <u>0</u> }

$$span\{\underline{u}\}\$$
 for some $\underline{u} \neq \underline{0}$ (a line thru the origin)

- $span\{\underline{u}, \underline{v}\}$ for some $\{\underline{u}, \underline{v}\}$ linearly independent 2 dimensional subspaces
- $span\{\underline{u}, \underline{v}, \underline{w}\} = \mathbb{R}^3$ for $\{\underline{u}, \underline{v}, \underline{w}\}$ linearly independent 3 dimensional (sub)space.
- We realized that what happens in \mathbb{R}^3 with respect to subspaces, generalizes to \mathbb{R}^n .

Towards the end of class on Friday we realized that for an $m \times n$ matrix A,

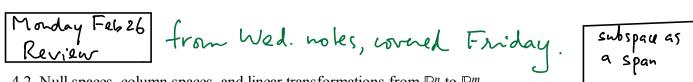
Nul $A := \{ \underline{x} \in \mathbb{R}^n \text{ for which } A \underline{x} = \underline{0} \}$

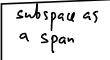
is a subspace. This is an <u>implicit way</u> to specify a subspace, because you're prescribing equations which the elements \underline{x} musts satisfy, but not explicitly saying what the elements are.

Picking up where we left off

Exercise 1a) For the same matrix A as in Exercise 2 from Wednesday's notes, express the vectors in Nul(A) explicitly, using the methods of Chapters 1-2. Notice these are vectors in the domain of the associated linear transformation $T : \mathbb{R}^5 \to \mathbb{R}^3$ given by $T(\underline{x}) = A \underline{x}$, so are a subspace of \mathbb{R}^5 .

$$A \stackrel{=}{\times} = \stackrel{=}{0} \qquad A = \begin{bmatrix} 1 & -2 & 0 & -1 & 1 \\ -2 & 4 & 0 & 2 & -2 \\ 3 & -6 & 4 & 1 & 7 \end{bmatrix} \stackrel{=}{0} \stackrel{=}{0} reduces to \begin{bmatrix} 1 & -2 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \stackrel{=}{0} \stackrel{=}$$





4.2 Null spaces, column spaces, and linear transformations from \mathbb{R}^n to \mathbb{R}^m .

<u>Definition</u> Let A be an $m \times n$ matrix, expressed in column form as $A = [\underline{a}_1 \ \underline{a}_2 \ \underline{a}_3 \ \dots \ \underline{a}_n]$ The column space of A, written as Col A, is the span of the columns:

$$Col A = span \{ \underline{a}_1 \ \underline{a}_2 \ \underline{a}_3 \ \dots \ \underline{a}_n \}.$$

Equivalently, since

$$A \underline{x} = x_1 \underline{a}_1 + x_2 \underline{a}_2 + \dots + x_n \underline{a}_n$$

we see that Col A is also the range of the linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ given by $T(\underline{x}) = A \underline{x}$, i.e.

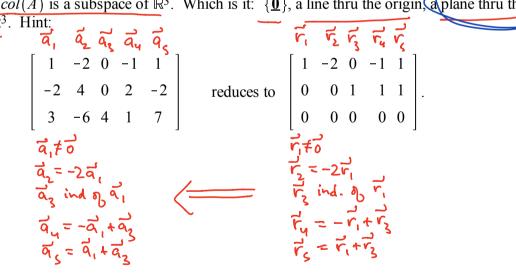
$$Col A = \{ \underline{b} \in \mathbb{R}^m \text{ such that } \underline{b} = A \underline{x} \text{ for some } \underline{x} \in \mathbb{R}^n \}.$$

<u>Theorem</u> By the "spans are subspaces" theorem, Col(A) is always a subspace of \mathbb{R}^m .

Exercise 2a) Consider

$$A = \begin{bmatrix} 1 & -2 & 0 & -1 & 1 \\ -2 & 4 & 0 & 2 & -2 \\ 3 & -6 & 4 & 1 & 7 \end{bmatrix}.$$

By the Theorem, col(A) is a subspace of \mathbb{R}^3 . Which is it: $\{\underline{0}\}$, a line thru the origin, a plane thru the origin, or all of \mathbb{R}^3 . Hint:



<u>2b</u>) Is there a more efficient way to express Col A as a span that doesn't require all five column vectors?

$$c_{1}\vec{a}_{1} + c_{2}\vec{a}_{2} + c_{3}\vec{a}_{3} + c_{4}\vec{a}_{4} + c_{5}\vec{a}_{5} = c_{1}\vec{a}_{1} + c_{2}(-2\vec{a}_{1}) + c_{3}\vec{a}_{3} + c_{4}(-\vec{a}_{1} + \vec{a}_{3}) + c_{5}(\vec{a}_{1} + \vec{a}_{3}) = d_{1}\vec{a}_{1} + d_{3}\vec{a}_{3} = Col A = span\{\vec{a}_{1}, \vec{a}_{2}, ...\vec{a}_{5}\}$$

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Not all bases are created equal!

<u>Theorem</u>: Let $span\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_p\} = H$ be a subspace. The following *elementary operations* do not effect the span of the resulting ordered set:

(i) swap two of the vectors in the set, i.e. replace \underline{v}_i with \underline{v}_k , and replace \underline{v}_k with \underline{v}_i .

$$span \{\vec{v}_{1}, ..., \vec{v}_{j}, ..., \vec{v}_{p}\} = span \{\vec{v}_{1}, ..., \vec{v}_{k}, ..., \vec{v}_{p}\} = span \{\vec{v}_{1}, ..., \vec{v}_{k}, ..., \vec{v}_{p}\} = span \{\vec{v}_{1}, ...$$

Exercise 1) Use the "change of spanning set" theorem above, to find a better basis for Col A then the one we came up with by culling dependent vectors, on Friday. Hint: Use elementary column operations to compute the reduced column echelon form of A. Illustrate why this new basis is a better basis for Col A by seeing how easy it is to express any one of the original column vectors in terms of this improved basis.

In this example, $A = [\underline{a}_1 \ \underline{a}_2 \ \underline{a}_3 \ \underline{a}_4 \ \underline{a}_5]$ and $Col A = span \{\underline{a}_1, \underline{a}_2, \underline{a}_3, \underline{a}_4, \underline{a}_5\} = span \{\overline{a}_1, \overline{a}_2\}$.

	1	-2	0	-1	1]
<i>A</i> =	-2	4	0	2	-2	
	3	-6	4	1	7	

As we just reviewed, on Friday we realized that a pretty good basis for Col A is $\{\underline{a}_1, \underline{a}_3\}$:

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$$\begin{bmatrix} 1 & -2 & 0 & -1 & 1 \\ -2 & 4 & 0 & 2 & -2 \\ 3 & -6 & 4 & 1 & 7 \end{bmatrix}$$
row reduces to
$$\begin{bmatrix} 1 & -2 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Now column reduce A to get a basis for Col A that's as good as you could hope for....and show this by expressing each of the original columns in terms of this basis.

, , , , .	$A = \begin{bmatrix} 1 & -2 & 0 & -1 & 1 \\ -2 & 4 & 0 & 2 & -2 \\ 3 & -6 & 4 & 1 & 7 \end{bmatrix}$	Col A basic
$rcef[$ $2\vec{a}_{1} + \vec{a}_{2} \rightarrow \vec{a}_{3}$ $\vec{a}_{1} + \vec{a}_{3} \rightarrow \vec{a}_{3}$ $\vec{a}_{1} + \vec{a}_{5} \rightarrow \vec{a}_{5}$	$ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 \\ 3 & 0 & 4 & 4 & 4 \end{bmatrix} $	$\vec{a}_{S} = 1 \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + 7 \begin{bmatrix} 1 \\ -2 \\ -2 \\ 7 \end{bmatrix}$
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-322+21-7 21	$ \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} $	