Fri Feb 23

- 4.2 - 4.3 nullspaces and column spaces; kernel and range of linear transformations as subspaces.

Linearly independent sets and bases for vector spaces.

Announcements:
- start in Wed. notes
- fft: (good lead in to HW).

Warm-up Exercise: The space of 2×2 matrices $M_{2\times 2} = \left\{ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} : a_{11}, a_{12}, a_{21}, a_{22} \in \mathbb{R} \right\}$

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is a vector space. Show that

$$H = \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}, a, b \in \mathbb{R} \right\} \text{ is a subspace of } M_{2\times 2}$$

Hint: you could show H contains the zero matrix and is closed under matrix addition & scalar multiplication, but it's quicker to show it's the span of two matrices.

Review
- vector space
- subspaces (sub vector spaces) H
  - a) $\vec{0} \in H$
  - b) $\vec{u}, \vec{v} \in H \Rightarrow \vec{u}+\vec{v} \in H$
  - c) $\vec{u} \in H, c \in \mathbb{R} \Rightarrow c\vec{u} \in H$
- if $H = \text{span}\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n\}$ then H is a subspace.

1) verify a), b), c)

a) $\vec{0} \in H$: $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in H$ $a=b=0$

b) $\left.\begin{matrix} a_{11} = a_{22} = a \\ a_{21} = 0 \end{matrix}\right\}$ condition to be in H

$\vec{u} = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$, $\vec{v} = \begin{bmatrix} c & d \\ 0 & c \end{bmatrix}$

$\vec{u}+\vec{v} = \begin{bmatrix} a+c & b+d \\ 0 & a+c \end{bmatrix} \in H$

$a_{11} = a_{22}$
$a_{21} = 0$

c) $\vec{u} = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$

$e\vec{u} = \begin{bmatrix} ea & eb \\ 0 & ea \end{bmatrix}$ $a_{11} = a_{22}$ ✓ $a_{21} = 0$ so $e\vec{u} \in H$

2) $H = \text{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$

$$\begin{bmatrix} a & b \\ 0 & a \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} + \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

from Wed. notes, covered Friday.

4.2 Null spaces, column spaces, and linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$.

Definition Let A be an $m \times n$ matrix, expressed in column form as $A = [\underline{\boldsymbol{a}}_1\ \underline{\boldsymbol{a}}_2\ \underline{\boldsymbol{a}}_3\ \dots\ \underline{\boldsymbol{a}}_n]$ The *column space* of A, written as $Col\ A$, is the span of the columns:

$$Col\ A = span\{\underline{\boldsymbol{a}}_1\ \underline{\boldsymbol{a}}_2\ \underline{\boldsymbol{a}}_3\ \dots\ \underline{\boldsymbol{a}}_n\}.$$

Equivalently, since

$$A\,\underline{\mathbf{x}} = x_1\,\underline{\boldsymbol{a}}_1 + x_2\,\underline{\boldsymbol{a}}_2 + \dots + x_n\,\underline{\boldsymbol{a}}_n$$

we see that $Col\ A$ is also the range of the linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ given by $T(\underline{\mathbf{x}}) = A\,\underline{\mathbf{x}}$, i.e

$$Col\ A = \{\underline{\boldsymbol{b}} \in \mathbb{R}^m \text{ such that } \underline{\boldsymbol{b}} = A\,\underline{\mathbf{x}} \text{ for some } \underline{\mathbf{x}} \in \mathbb{R}^n\}.$$

Theorem By the "spans are subspaces" theorem, $Col(A)$ is always a subspace of $\mathbb{R}^m$.

Exercise 2a) Consider

$$A = \begin{bmatrix} 1 & -2 & 0 & -1 & 1 \\ -2 & 4 & 0 & 2 & -2 \\ 3 & -6 & 4 & 1 & 7 \end{bmatrix}.$$

By the Theorem, $col(A)$ is a subspace of $\mathbb{R}^3$. Which is it: $\{\underline{\mathbf{0}}\}$, a line thru the origin, a plane thru the origin, or all of $\mathbb{R}^3$. Hint:

$$\begin{array}{ccccc} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 & \vec{a}_4 & \vec{a}_5 \end{array} \qquad\qquad \begin{array}{ccccc} \vec{r}_1 & \vec{r}_2 & \vec{r}_3 & \vec{r}_4 & \vec{r}_5 \end{array}$$

$$\begin{bmatrix} 1 & -2 & 0 & -1 & 1 \\ -2 & 4 & 0 & 2 & -2 \\ 3 & -6 & 4 & 1 & 7 \end{bmatrix} \text{ reduces to } \begin{bmatrix} 1 & -2 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

$\vec{a}_1 \neq \vec{0}$
$\vec{a}_2 = -2\vec{a}_1$
$\vec{a}_3$ ind of $\vec{a}_1$
$\vec{a}_4 = -\vec{a}_1 + \vec{a}_3$
$\vec{a}_5 = \vec{a}_1 + \vec{a}_3$

$\Longleftarrow$

$\vec{r}_1 \neq \vec{0}$
$\vec{r}_2 = -2\vec{r}_1$
$\vec{r}_3$ ind. of $\vec{r}_1$
$\vec{r}_4 = -\vec{r}_1 + \vec{r}_3$
$\vec{r}_5 = \vec{r}_1 + \vec{r}_3$

2b) Is there a more efficient way to express $Col\ A$ as a span that doesn't require all five column vectors?

YES.

$$c_1\vec{a}_1 + c_2\vec{a}_2 + c_3\vec{a}_3 + c_4\vec{a}_4 + c_5\vec{a}_5$$
$$= c_1\vec{a}_1 + c_2(-2\vec{a}_1) + c_3\vec{a}_3 + c_4(-\vec{a}_1 + \vec{a}_3) + c_5(\vec{a}_1 + \vec{a}_3)$$
$$= d_1\vec{a}_1 + d_3\vec{a}_3$$

$$Col\ A = span\{\vec{a}_1, \vec{a}_2, \dots \vec{a}_5\} = span\{\vec{a}_1, \vec{a}_3\}$$

so, col A was just a plane thru origin

from Wed notes, covered Friday

<u>Definition</u>: If a set of vectors $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n\}$ in a vector space V is linearly independent and also spans V, then the collection is called a *basis* for V.

<u>Exercise 3</u> Exhibit a basis for *col A* in <u>Exercise 2</u>.

$$\{\vec{a}_1, \vec{a}_3\} = \left\{ \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} \right\}.$$

<u>Exercise 4</u> Exhibit a basis for P_3 in <u>Exercise 1</u>

$$\{a_0 + a_1 t + a_2 t^2 + a_3 t^3\} = \text{span}\{1, t, t^2, t^3\}$$

and we showed $\{1, t, t^2, t^3\}$ are lin-ind., so a basis

We've seen that one (explicit) way that subspaces arise is as the span of a specified collection of vectors. The primary (implicit) way that subspaces are described is related to the following:

Definition: The *null space* of an $m \times n$ matrix A is the set of $\underline{\mathbf{x}} \in \mathbb{R}^n$ for which $A\,\underline{\mathbf{x}} = \underline{\mathbf{0}}$. We denote this set by *Nul A*. Equivalently, in terms of the associated linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ given by $T(\underline{\mathbf{x}}) = A\,\underline{\mathbf{x}}$, *Nul A* is the set of points in the domain which are transformed into the zero vector in the codomain.

Theorem Let A be an $m \times n$ matrix. Then *Nul A* is a subspace of $\mathbb{R}^n$.

$H = \{\vec{x} \in \mathbb{R}^n \text{ s.t. } A\vec{x} = \vec{0}\}$

proof: We need to check that for $H = Nul(A)$:

a) The zero vector of V is in H

is $\vec{0} \in H$: $A\vec{0} = \vec{0}$ so $\vec{0} \in H$.

b) H is closed under vector addition, i.e. for each $\underline{\mathbf{u}} \in H$, $\underline{\mathbf{v}} \in H$ *then* $\underline{\mathbf{u}} + \underline{\mathbf{v}} \in H$.

$\Downarrow$

$A\vec{u} = \vec{0}$, $A\vec{v} = \vec{0}$

so $A(\vec{u}+\vec{v}) = A\vec{u} + A\vec{v} = \vec{0} + \vec{0} = \vec{0}$

so $\vec{u}+\vec{v} \in H = \text{Nul } A$

c) H is closed under scalar multiplication, i.e for each $\underline{\mathbf{u}} \in H$, $c \in \mathbb{R}$, *then also* $c\,\underline{\mathbf{u}} \in H$.

$H = \text{Nul } A$

$\Downarrow$

$A\vec{u} = \vec{0}$.

$\Downarrow$

$A(c\vec{u}) = cA\vec{u} = \vec{0}$

so $c\vec{u} \in \text{Nul } A$

Exercise 1a) For the same matrix A as in Exercise 2 from Wednesday's notes, express the vectors in $Nul(A)$ explicitly, using the methods of Chapters 1-2. Notice these are vectors in the domain of the associated linear transformation $T:\mathbb{R}^5 \rightarrow \mathbb{R}^3$ given by $T(\underline{\boldsymbol{x}}) = A\,\underline{\boldsymbol{x}}$, so are a subspace of $\mathbb{R}^5$.

$A\vec{x}=\vec{0}$

$$A = \left[\begin{array}{ccccc|c} 1 & -2 & 0 & -1 & 1 & 0 \\ -2 & 4 & 0 & 2 & -2 & 0 \\ 3 & -6 & 4 & 1 & 7 & 0 \end{array}\right] \text{ reduces to } \left[\begin{array}{ccccc|c} 1 & -2 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \leftarrow$$

$\{\vec{x} : A\vec{x}=\vec{0}\}$

implicit description

$x_1 = 2t_2 + t_4 - t_5$
$x_2 = t_2$ free
$x_3 = -t_4 - t_5$
$x_4 = t_4$ free
$x_5 = t_5$ free

explicit description
$$\begin{bmatrix} x_1\\x_2\\x_3\\x_4\\x_5 \end{bmatrix} = t_2\begin{bmatrix}2\\1\\0\\0\\0\end{bmatrix} + t_4\begin{bmatrix}1\\0\\-1\\1\\0\end{bmatrix} + t_5\begin{bmatrix}-1\\0\\-1\\0\\1\end{bmatrix}$$

1b) Exhibit a basis for $Nul(A)$.

$$\text{Nul } A = \text{span}\left\{\begin{bmatrix}2\\1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\-1\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\-1\\0\\1\end{bmatrix}\right\}$$

actually a basis

The ideas of nullspace and column space generalize to arbitrary linear transformations between vectors spaces - with slightly more general terminology.

Definition Let V and W be vector spaces. A function $T: V \to W$ is called a *linear transformation* if for each $\underline{\mathbf{x}} \in V$ there is a unique vector $T(\underline{\mathbf{x}}) \in W$ and so that

(i) $T(\underline{\mathbf{u}} + \underline{\mathbf{v}}) = T(\underline{\mathbf{u}}) + T(\underline{\mathbf{v}})$ for all $\underline{\mathbf{u}}, \underline{\mathbf{v}} \in V$

(ii) $T(c\,\underline{\mathbf{u}}) = c\,T(\underline{\mathbf{u}})$ for all $\underline{\mathbf{u}} \in V$, $c \in \mathbb{R}$

Definition The *kernel* (or *nullspace*) of T is defined to be $\{\underline{\mathbf{u}} \in V:\ T(\underline{\mathbf{u}}) = \underline{\mathbf{0}}\}$.

Definition The *range* of T is $\{\underline{\mathbf{w}} \in W:\ \underline{\mathbf{w}} = T(\underline{\mathbf{v}})\ \textit{for some}\ \underline{\mathbf{v}} \in V\}$.

Theorem Let $T: V \to W$ be a *linear transformation.* Then the kernel of T is a subspace of V. The range of T is a subspace of W.

Remark: The theorem generalizes our earlier one about *Nul A* and *Col A*, for matrix transformations $T: \mathbb{R}^n \to \mathbb{R}^m$, $T(\underline{\mathbf{x}}) = A\,\underline{\mathbf{x}}$.