think sub-vector-space

Definition: A *subspace* of a vector space V is a subset H of V which is itself a vector space with respect to the addition and scalar multiplication in V. As soon as one verifies a), b), c) below for H, it will be a subspace, because H will "inherit" the other axioms just by being contained in V.

a) The zero vector of V is in H (property 4)

b) H is closed under vector addition, i.e. for each $\underline{\boldsymbol{u}} \in H, \underline{\boldsymbol{v}} \in H$ then $\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}} \in H$. (property 1)

c) H is closed under scalar multiplication, i.e for each $\underline{\boldsymbol{u}} \in H, c \in \mathbb{R}$, then also $c\,\underline{\boldsymbol{u}} \in H$. (property 6)

(c) $\Rightarrow$ (a)
$0\vec{u} = \vec{0}$

Just to double check that the other properties get inherited:
Definition A *vector space* is a nonempty set V of objects, called *vectors*, on which are defined two operations, called *addition* and *scalar multiplication,* so that the ten axioms listed below hold. These axioms must hold for all vectors $\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}}, \underline{\boldsymbol{w}}$ in V, and for all scalars $c, d \in \mathbb{R}$.

(b) 1. The sum of $\underline{\boldsymbol{u}}$ and $\underline{\boldsymbol{v}}$, denoted by $\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}}$, is (also) in V (closure under addition.)

2. $\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}} = \underline{\boldsymbol{v}} + \underline{\boldsymbol{u}}$ (commutative property of addition)

3. $(\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}}) + \underline{\boldsymbol{w}} = \underline{\boldsymbol{u}} + (\underline{\boldsymbol{v}} + \underline{\boldsymbol{w}})$ (associative property of addition)

(a) 4. There is a **zero** vector $\underline{\mathbf{0}}$ in V so that $\underline{\boldsymbol{u}} + \underline{\mathbf{0}} = \underline{\boldsymbol{u}}$. (additive identity)

13 & (c) 5. For each $\underline{\boldsymbol{u}} \in V$ there is a vector $-\underline{\boldsymbol{u}} \in V$ so that $\underline{\boldsymbol{u}} + (-\underline{\boldsymbol{u}}) = \underline{\mathbf{0}}$. (additive inverses)

(c) 6. The scalar multiple of $\vec{u}$ by c, denoted by $c\,\vec{u}$ is (also) in V. (closure under scalar multiplication)

7. $c(\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}}) = c\,\underline{\boldsymbol{u}} + c\,\underline{\boldsymbol{v}}$ (scalar multiplication distributes over vector addition)

8. $(c + d)\,\underline{\boldsymbol{u}} = c\,\underline{\boldsymbol{u}} + d\,\underline{\boldsymbol{u}}$. (scalar multiplication distributes over scalar addition)

9. $c\,(d\,\underline{\boldsymbol{u}}) = (c\,d)\,\underline{\boldsymbol{u}}$ (associative property of scalar multiplication)

10. $1\,\underline{\boldsymbol{u}} = \underline{\boldsymbol{u}}$ (multiplicative identity)

The following three algebra rules follow from the first 10, and are also useful:

11) $0\,\underline{\boldsymbol{u}} = \vec{0}$

12) $c\,\underline{\mathbf{0}} = \underline{\mathbf{0}}$.

13) $-\underline{\boldsymbol{u}} = (-1)\,\underline{\boldsymbol{u}}$.

Big Exercise: The vector space $\mathbb{R}^n$ has subspaces! But there aren't very many kinds, it turns out. (Even though there are countless kinds of *subsets* of $\mathbb{R}^n$.) Let's find *all* the possible kinds of subspaces of $\mathbb{R}^3$, using our expertise with matrix reduced row echelon form.

All sub (vector) spaces of $\mathbb{R}^3$. From small to large.

Let H be a subspace of $\mathbb{R}^3$

(a) $\vec{0} \in H$

(b) $\vec{u}, \vec{v} \in H \Rightarrow \vec{u} + \vec{v} \in H$

(c) $\vec{u} \in H \Rightarrow c\vec{u} \in H$ for all $c \in \mathbb{R}$.

- $\vec{0} \in H$ by (a)
- $\longrightarrow$ H could be $\{\vec{0}\} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

if more to H, let $\vec{u} \in H$, $\vec{u} \neq \vec{0}$ $\quad \vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$

by (c) $\text{span}\{\vec{u}\} = \{c\vec{u} \text{ s.t. } c \in \mathbb{R}\}$ is contained in H

note (a), (b), (c) hold for $\text{span}\{\vec{u}\}$

- $\longrightarrow$ so H could be $\text{span}\{\vec{u}\}$ $=$ line thru $\vec{0}$, with direction vector $\vec{u}$.

If there's more to H, let $\vec{v} \in H$, $\vec{v} \neq c\vec{u}$ for any c

(c) says $\text{span}\{\vec{v}\} \subset H$ (subset)

- $\xrightarrow[\text{H could be}]{}$ (b) $\Rightarrow$ $\underbrace{\text{span}\{\vec{u}, \vec{v}\}}_{\text{is a subspace}} \subset H$ $\quad$ ($c_1\vec{u} \in H$, $c_2\vec{v} \in H \Rightarrow c_1\vec{u} + c_2\vec{v} \in H$

is a subspace:

(a) $\vec{0} = 0\vec{u} \in H$

(b) $(c_1\vec{u} + c_2\vec{v}) + (d_1\vec{u} + d_2\vec{v}) = (c_1+d_1)\vec{u} + (c_2+d_2)\vec{v} \in H$

(c) $c(c_1\vec{u} + c_2\vec{v}) = cc_1\vec{u} + cc_2\vec{v} \in H$

plane thru $\vec{0}$.

$$\begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \\ u_3 & v_3 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\searrow \begin{bmatrix} 1 & * \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ says } \vec{v} = *\vec{u}$$

- There could be more than just the $\text{span}\{\vec{u}, \vec{v}\}$ in H.

So, there is $\vec{w} \in H$, $\vec{w} \notin \text{span}\{\vec{u}, \vec{v}\}$

Since H is closed under $+$ & scalar multiplication deduce $\text{span}\{\vec{u}, \vec{v}, \vec{w}\}$ is in H.

$\text{span}\{\vec{u}, \vec{v}, \vec{w}\} = \mathbb{R}^3$

because

$$\begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 0 \end{bmatrix}$$

would say that

$\vec{w} = a\vec{u} + b\vec{v}$

This can't happen because we assumed $\vec{w}$ was NOT in span$\{\vec{u}, \vec{v}\}$.

since $[\vec{u}\ \vec{v}\ \vec{w}]$ reduces to I.

we can always (uniquely solve)

$$[\vec{u}\ \vec{v}\ \vec{w}]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \vec{b} \quad \text{for } \vec{x} \quad (\text{for every } \vec{b} \in \mathbb{R}^3)$$

i.e. $\text{span}\{\vec{u}, \vec{v}, \vec{w}\} = \mathbb{R}^3$

Wed Feb 21

- 4.1-4.2 Vector spaces and subspaces; null spaces, column spaces, and the connections to linear transformations

Announcements:
- Homework for next week includes
  4.1 1, 3, 5, 7, 9, 13, 19, 21, 23, 31
  4.2 1, 3, 7, 9, 15, 17, 21, 25, 27, 31, 33
- Quiz today

Warm-up Exercise: The matrix
'til 12:57

$$\begin{array}{ccc} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \\ \end{array} \begin{bmatrix} 1 & 2 & 1 \\ -2 & 1 & -7 \\ 3 & 1 & 8 \end{bmatrix} \text{ reduces to } \begin{array}{ccc} \vec{r}_1 & \vec{r}_2 & \vec{r}_3 \end{array} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

express $\begin{bmatrix} 1 \\ -7 \\ 8 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$.

$$= 3\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} - 1\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

review

column depencies of a matrix correspond to solutions $\vec{x}$ to $A\vec{x} = \vec{0}$
these stay the same as you reduce the matrix:

$$\vec{r}_3 = 3\vec{r}_1 - \vec{r}_2 \qquad \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

So $\vec{a}_3 = 3\vec{a}_1 - \vec{a}_2$

(or $\vec{0} = 3\vec{a}_1 - \vec{a}_2 - \vec{a}_3$

$$A\begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix} = \vec{0}$$

We've been discussing the abstract notions of *vector spaces* and *subspaces,* with some specific examples to help us with our intuition. Today we continue that discussion. We'll continue to use exactly the same language we used in Chapters 1-2 .... except now it's for general vector spaces:

Let V be a *vector space* (Do you recall that definition, at least roughly speaking?)

<u>Definition:</u> If we have a collection of p vectors $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_p\}$ in V, then any vector $\underline{\boldsymbol{v}} \in V$ that can be expressed as a sum of scalar multiples of these vectors is called a *linear combination* of them. In other words, if we can write

$$\underline{\boldsymbol{v}} = c_1\underline{\boldsymbol{v}}_1 + c_2\underline{\boldsymbol{v}}_2 + \ldots + c_p\underline{\boldsymbol{v}}_p ,$$

then $\underline{\boldsymbol{v}}$ is a *linear combination* of $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_p$. The scalars $c_1, c_2, \ldots, c_p$ are called the *linear combination coefficients* or *weights*.

<u>Definition</u> The *span* of a collection of vectors, written as $span\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_p\}$, is the collection of all linear combinations of those vectors.

<u>Definition:</u>

<u>a)</u> An indexed set of vectors $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_p\}$ in V is said to be *linearly independent* if no one of the vectors is a linear combination of (some) of the other vectors. The concise way to say this is that the only way $\underline{\mathbf{0}}$ can be expressed as a linear combination of these vectors,

$$c_1\underline{\boldsymbol{v}}_1 + c_2\underline{\boldsymbol{v}}_2 + \ldots + c_p\underline{\boldsymbol{v}}_p = \underline{\mathbf{0}} ,$$

is for all of the weights $c_1 = c_2 = \ldots = c_p = 0$.

<u>b)</u> An indexed set of vectors $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_p\}$ is said to be *linearly dependent* if at least one of these vectors is a linear combination of (some) of the other vectors. The concise way to say this is that there *is* some way to write $\underline{\mathbf{0}}$ as a linear combination of these vectors

$$c_1\underline{\boldsymbol{v}}_1 + c_2\underline{\boldsymbol{v}}_2 + \ldots + c_p\underline{\boldsymbol{v}}_p = \underline{\mathbf{0}}$$

where *not all* of the $c_j = 0$. (We call such an equation a *linear dependency.* Note that if we have any such linear dependency, then any $\underline{\boldsymbol{v}}_j$ with $c_j \neq 0$ is a linear combination of the remaining $\underline{\boldsymbol{v}}_k$ with $k \neq j$. We say that such a $\underline{\boldsymbol{v}}_j$ is *linearly dependent* on the remaining $\underline{\boldsymbol{v}}_k$.)

And from yesterday,

Definition: A *subspace* of a vector space V is a subset H of V which is itself a vector space with respect to the addition and scalar multiplication in V. As soon as one verifies a), b), c) below for H, it will be a subspace.

a) The zero vector of V is in H

b) H is closed under vector addition, i.e. for each $\underline{\boldsymbol{u}} \in H, \underline{\boldsymbol{v}} \in H$ then $\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}} \in H$.

c) H is closed under scalar multiplication, i.e for each $\underline{\boldsymbol{u}} \in H$, $c \in \mathbb{R}$, then also $c\,\underline{\boldsymbol{u}} \in H$.

Theorem (spans are subspaces) Let V be a vector space, and let $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots\ \underline{\boldsymbol{v}}_n\}$ be a set of vectors in V.
Then $H = span\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots\ \underline{\boldsymbol{v}}_n\}$ is a subspace of V.
proof: We need to check that for $H = span\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots\ \underline{\boldsymbol{v}}_n\}$

a) The zero vector of V is in H

$\vec{0} = 0\vec{v}_1 \in H$ (11 in vector space axioms)

b) H is closed under vector addition, i.e. for each $\underline{\boldsymbol{u}} \in H, \underline{\boldsymbol{v}} \in H$ then $\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}} \in H$.

Let $\vec{u}, \vec{v} \in span\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n\}$

So $\vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$

$\vec{v} = d_1\vec{v}_1 + d_2\vec{v}_2 + \cdots + d_n\vec{v}_n$

don't need, because addition is associative

so $\vec{u}+\vec{v} = (c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n) + (d_1\vec{v}_1 + d_2\vec{v}_2 + \cdots + d_n\vec{v}_n)$

$= (c_1\vec{v}_1 + d_1\vec{v}_1) + (c_2\vec{v}_2 + d_2\vec{v}_2) + \cdots + (c_n\vec{v}_n + d_n\vec{v}_n)$ ~ addition is commutative

c) H is closed under scalar multiplication, i.e for each $\underline{\boldsymbol{u}} \in H$, $c \in \mathbb{R}$, then also $c\,\underline{\boldsymbol{u}} \in H$.

$= (c_1 + d_1)\vec{v}_1 + (c_2 + d_2)\vec{v}_2 + \cdots + (c_n + d_n)\vec{v}_n$

scalar addition distributes over scalar mult.

So $\vec{u}+\vec{v} \in H$

$c(c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n) = cc_1\vec{v}_1 + cc_2\vec{v}_2 + \cdots + cc_n\vec{v}_n$

property (7)

so $c\vec{u} \in H$

Remark Using minimal spanning sets was how we were able to characterize all possible subspace of $\mathbb{R}^3$ yesterday (or today, if we didn't finish on Tuesday). Can you characterize all possible subsets of $\mathbb{R}^n$ in this way?

Example: Let P_n be the space of polynomials of degree at most n,

$$P_n = \{p(t) = a_0 + a_1 t + a_2 t^2 + ... + a_n t^n \text{ such that } a_0, a_1, ... a_n \in \mathbb{R}\}$$

Note that P_n is the span of the $(n + 1)$ functions

$$p_0(t) = 1, p_1(t) = t, p_2(t) = t^2, ... \ p_n(t) = t^n.$$

Although we often consider P_n as a vector space on its own, we can also consider it to be a subspace of the much larger vector space V of all functions from $\mathbb{R}$ to $\mathbb{R}$.

Exercise 1 abbreviating the functions by their formulas, we have

$$P_3 = span\{ 1, t, t^2, t^3 \}.$$

Are the functions in the set $\{ 1, t, t^2, t^3 \}$ linearly independent or linearly dependent?.

assuming domain is $\mathbb{R}$ for these functions.

dependency equation

$$c_1 \cdot 1 + c_2 t + c_3 t^2 + c_4 t^3 \equiv 0 \quad (\forall t)$$

by fund. thm of algebra a cubic (or lower degree) polynomial as at most 3 roots. (unless it's the zero polynomial)

since $\mathbb{R}$ has more than 3 points we deduce we have the zero polynomial, i.e. $c_1 = c_2 = c_3 = c_4 = 0$

better: If $c_1 + c_2 t + c_3 t^2 + c_4 t^3 \equiv 0 \quad \Rightarrow c_1 = 0$

$\frac{d}{dt}$: $c_2 + c_3 2t + c_4 3t^2 \equiv \frac{d}{dt} 0 \equiv 0 \quad \Rightarrow c_2 = 0$

$\frac{d}{dt}$: $2c_3 + 6c_4 t \equiv 0 \quad \Rightarrow c_3 = 0$

$\frac{d}{dt}$: $6c_4 = 0 \quad \longrightarrow c_4 = 0$