Fri Feb 2

- 2.2 matrix inverses

Announcements:

- T/W notes.
  (Monday - Tuesday 2.2-2.3)
- all HW & quizzes should be graded & in "Return" folder.
- goal: FFT's returned Monday
- exam 1 next Friday. Feb 9

Warm-up Exercise: Compute

'til 12:58

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 4 \end{bmatrix}}_{B}\underbrace{\begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 1 & 0 \end{bmatrix}}_{A} = \begin{bmatrix} 4 & 0 \\ 3 & \boxed{-2} \end{bmatrix}$$

$1\cdot 2 - 2\cdot 1 + 3\cdot 0 = 0$

$-1\cdot 2 + 0\cdot(-1) + 4\cdot 0 = \boxed{-2}$

and

$$\begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 4 \end{bmatrix} = \begin{bmatrix} -1 & 2 & 11 \\ \boxed{1} & 0 & -4 \\ \boxed{1} & 2 & 3 \end{bmatrix}$$

$$B_{m\times n}A_{n\times p} = [BA]_{mp}$$

- $col_j(BA) = B\, col_j(A)$ $\qquad BA \neq AB$
- $entry_{ij}(BA) = row_i(B)\cdot col_j(A)$

Exercise 2   Compute

$$\underbrace{\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}}_{B}\underbrace{\begin{bmatrix} 0 & 1 & -2 & 3 \\ 1 & 0 & 4 & 1 \end{bmatrix}}_{A} = \begin{bmatrix} 2 & 1 & 6 & 5 \\ 3 & -1 & 14 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0\begin{bmatrix} 1 \\ -1 \end{bmatrix} + 1\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

or $\begin{bmatrix} 0\cdot 1 + 1\cdot 2 \\ -1\cdot 0 + 3\cdot 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

(From Tuesday notes)

After Friday warmup, we did Exercise 3, to re-enforce idea that matrix product is matrix for composition of transformations

$\to$ 3rd col. $\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} -2 \\ 4 \end{bmatrix} = \begin{bmatrix} -2+8 \\ 2+12 \end{bmatrix}$

Exercise 3 For $T_1 : \mathbb{R}^4 \to \mathbb{R}^2$ $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$

$$T_1(\underline{\mathbf{x}}) = \underbrace{\begin{bmatrix} 0 & 1 & -2 & 3 \\ 1 & 0 & 4 & 1 \end{bmatrix}}_{A}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \qquad T_2(\underline{\mathbf{y}}) = \underbrace{\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}}_{B}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

compute $T_2\left(T_1(\underline{\mathbf{x}})\right)$. How does this computation relate to Exercise 2?

$$T_2(T_1(\vec{x})) = T_2(A\vec{x}) = T_2\left(\begin{bmatrix} x_2 - 2x_3 + 3x_4 \\ x_1 + 4x_3 + x_4 \end{bmatrix}\right)$$

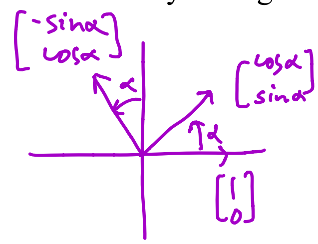
$$= \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} x_2 - 2x_3 + 3x_4 \\ x_1 + 4x_3 + x_4 \end{bmatrix}$$

$$= \begin{bmatrix} 1\cdot(x_2 - 2x_3 + 3x_4) + 2(x_1 + 4x_3 + x_4) \\ -1(x_2 - 2x_3 + 3x_4) + 3(x_1 + 4x_3 + x_4) \end{bmatrix}$$

$$= \begin{bmatrix} 2x_1 + x_2 + 6x_3 + 5x_4 \\ 3x_1 - x_2 + 14x_3 + 0x_4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 6 & 5 \\ 3 & -1 & 14 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Exercise 4 Recall from Monday that the linear transformation which rotates counterclockwise by an angle α has matrix



$$[Rot_\alpha] = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}$$

and

$$[Rot_\beta] = \begin{bmatrix} \cos(\beta) & -\sin(\beta) \\ \sin(\beta) & \cos(\beta) \end{bmatrix}$$

use this in Friday FFT

Compute the product

$$[Rot_\beta][Rot_\alpha].$$

What do you see?

Then (on Friday) moved to matrix algebra from Wed. notes

2.1 matrix algebra....we already talked about matrix multiplication. It interacts with matrix addition in interesting ways. We can also add and scalar multiply matrices of the same size, just treating them as oddly-shaped vectors:

Matrix algebra:

- addition and scalar multiplication: Let $A_{m \times n}$, $B_{m \times n}$ be two matrices of the same dimensions (m rows and n columns). Let $entry_{i\,j}(A) = a_{i\,j}$, $entry_{i\,j}(B) = b_{i\,j}$. (In this case we write $A = \left[a_{i\,j}\right]$, $B = \left[b_{i\,j}\right]$.) Let c be a scalar. Then

$$entry_{i\,j}(A+B) := a_{i\,j} + b_{i\,j}.$$
$$entry_{i\,j}(c\,A) := c\,a_{i\,j}.$$

In other words, addition and scalar multiplication are defined analogously as for vectors. In fact, for these two operations you can just think of matrices as vectors written in a rectangular rather than row or column format.

Exercise 1) Let $A := \begin{bmatrix} 1 & -2 \\ 3 & -1 \\ 0 & 3 \end{bmatrix}$ and $B := \begin{bmatrix} 0 & 27 \\ 5 & -1 \\ -1 & 1 \end{bmatrix}$. Compute $4\,A - B$.

$$4A - B = 4\begin{bmatrix} 1 & -2 \\ 3 & -1 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} 0 & 27 \\ 5 & -1 \\ -1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 4 & -8 \\ 12 & -4 \\ 0 & 12 \end{bmatrix} - \begin{bmatrix} 0 & 27 \\ 5 & -1 \\ -1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 4 & -35 \\ 7 & -3 \\ 1 & 11 \end{bmatrix}$$

vector properties of matrix addition and scalar multiplication

But other properties you're used to do hold:

- $+$ is commutative $\qquad A + B = B + A$

$$entry_{ij}(A+B) = a_{ij} + b_{ij} = b_{ij} + a_{ij} = entry_{ij}(B+A)$$

- $+$ is associative $\qquad (A+B)+C = A+(B+C)$

the ij entry of each side is $a_{ij} + b_{ij} + c_{ij}$

- scalar multiplication distributes over $+$ $\qquad c(A+B) = cA + cB$.

ij entry of LHS is $c\left(a_{ij} + b_{ij}\right) = c\left(a_{ij} + b_{ij}\right) = ij$ *entry of RHS*

More interesting are how matrix multiplication and addition interact:

Check some of the following. Let I_n be the $n \times n$ identity matrix, with $I_n \underline{\boldsymbol{x}} = \underline{\boldsymbol{x}}$ for all $\underline{\boldsymbol{x}} \in \mathbb{R}^n$. Let A, B, C have compatible dimensions so that the indicated expressions make sense. Then

a $A(BC) = (AB)C$ (associative property of multiplication)

$A_{m\times n}(B_{n\times p}\, C_{pq})$ $\quad$ $(A_{m\times n} B_{n\times p})C_{pq}$

$\underbrace{A_{m\times n}(BC)_{n\times q}}_{[A(BC)]_{m\times q}}$ $\quad$ $(AB)_{m\times p} C_{p\times q}$

$[(AB)C]_{m\times q}$

look at j^{th} columns :

$$\begin{aligned} col_j(A(BC)) &= A\, col_j(BC) \\ &= A(B\, col_j C) \\ &= (AB)\, col_j C \\ &= col_j[(AB)C] \end{aligned}$$

because of composition rule!!

b $A(B + C) = AB + AC$ (left distributive law)

c $(A + B)C = AC + BC$ (right distributive law)

d $r\,AB = (rA)B = A(rB)$ for any scalar r.

$A_{m\times n} I_{n\times n}$

e) If $A_{m \times n}$ then $I_m A = A$ and $A I_n = A$.

Warning: $AB \neq BA$ in general. In fact, the sizes won't even match up if you don't use square matrices.