Theorem: Let $A_{n \times n}$. Then A^{-1} exists if and only if $det(A) \neq 0$.

proof: We already know that A^{-1} exists if and only if the reduced row echelon form of A is the identity matrix. Now, consider reducing A to its reduced row echelon form, and keep track of how the determinants of the corresponding matrices change: As we do elementary row operations,

- if we swap rows, the sign of the determinant switches.
- if we factor non-zero factors out of rows, we factor the same factors out of the determinants.
- if we replace a row by its sum with a multiple of another row, the determinant is unchanged.

Thus,

$$|A| = c_1|A_1| = c_1c_2|A_2| = \ldots = c_1c_2 \ldots c_N\left|rref(A)\right|$$

where the nonzero c_k's arise from the three types of elementary row operations. If $rref(A) = I$ its determinant is 1, and $|A| = c_1c_2 \ldots c_N \neq 0$. If $rref(A) \neq I$ then its bottom row is all zeroes and its determinant is zero, so $|A| = c_1c_2 \ldots c_N(0) = 0$. Thus $|A| \neq 0$ if and only if $rref(A) = I$ if and only if A^{-1} exists !

Theorem: Using the same ideas as above, we can show that $det(A\,B) = det(A)det(B)$. This is an important identity that gets used, for example, in multivariable change of variables formulas for integration, using the Jacobian matrix. (It is not true that $det(A + B) = det(A) + det(B)$.)

Here's how to show $det(A\,B) = det(A)det(B)$: The key point is that if you do an elementary row operation to AB, that's the same as doing the elementary row operation to A, and then multiplying by B. With that in mind, if you do exactly the same elementary row operations as you did for A in the theorem above, you get

$$|A\,B| = c_1|A_1B| = c_1c_2|A_2B| = \ldots = c_1c_2 \ldots c_N\left|rref(A)B\right|.$$

If $rref(A) = I$, then from the theorem above, $|A| = c_1c_2 \ldots c_N$, and we deduce $|A\,B| = |A||B|$. If $rref(A) \neq I$, then its bottom row is zeroes, and so is the bottom row of $rref(A)B$. Thus $|A\,B| = 0$ and also $|A||B| = 0$.

Fri Feb 14

- 3.3 adjoint formula for inverses, Cramer's rule, geometric meanings of determinants.

Announcements: for last 25 minutes fft geometric meaning of determinants due Tues.

Warm-up Exercise: Compute the following two determinants by expanding across the 2nd row.

'til 12:57

Notice that

1) The cofactors are the same. because you crossed out row 2

2) You know that one of the determinants has to be zero

$$\begin{vmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{vmatrix} = -0\begin{vmatrix} 2 & -1 \\ -2 & 1 \end{vmatrix} + 3\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 2 \\ 2 & -2 \end{vmatrix} = 0 + 3\cdot 3 - 1(-6) = \underline{\underline{15}}$$

$$\begin{vmatrix} 1 & 2 & -1 \\ 1 & 2 & -1 \\ 2 & -2 & 1 \end{vmatrix} = 0 = -1\begin{vmatrix} 2 & -1 \\ -2 & 1 \end{vmatrix} + 2\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} + 1\begin{vmatrix} 1 & 2 \\ 2 & -2 \end{vmatrix}$$

2 rows are the same

$$= -1(0) + 2(3) + 1(-6) = 0!$$

Theorem: Let $A_{n \times n}$, and denote its cofactor matrix by $cof(A) = [C_{ij}]$, with $C_{ij} = (-1)^{i+j} M_{ij}$, and $M_{ij} =$ the determinant of the $(n-1) \times (n-1)$ matrix obtained by deleting row i and column j from A.

formal

Define the adjoint matrix to be the transpose of the cofactor matrix:

$$\underline{Adj(A)} := \underline{cof(A)}^T$$

Then, when A^{-1} exists it is given by the formula

$$A^{-1} = \frac{1}{det(A)} Adj(A)\,.$$

Exercise 1) Show that in the 2 × 2 case this reproduces the formula

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

$$\begin{matrix} + & - \\ - & + \end{matrix}$$

$$cof(A) = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$$

$$Adj(A) = cof(A)^T = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A^{-1} = \frac{1}{det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \checkmark$$

Example) For our friend $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{bmatrix}$ we worked out $cof(A) = \begin{bmatrix} 5 & 2 & -6 \\ 0 & 3 & 6 \\ 5 & -1 & 3 \end{bmatrix}$ so

$5 = +\begin{vmatrix} 3 & 1 \\ -2 & 1 \end{vmatrix}$,

$2 = -\begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix}$

etc.

transpose.

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{bmatrix} \quad adj(A) = \begin{bmatrix} 5 & 0 & 5 \\ 2 & 3 & -1 \\ -6 & 6 & 3 \end{bmatrix} \quad det(A) = 15,$$

so

$A \cdot Adj(A) = \begin{bmatrix} 15 & & \\ & & \\ & & \end{bmatrix}$

$$A^{-1} = \frac{1}{15}\begin{bmatrix} 5 & 0 & 5 \\ 2 & 3 & -1 \\ -6 & 6 & 3 \end{bmatrix}$$

Let's understand why the magic worked:

Exercise 2) Continuing with our example,

2a) The $(1, 1)$ entry of $(A)(Adj(A))$ is $15 = 1\cdot5 + 2\cdot2 + (-1)(-6)$. Explain why this is $det(A)$, expanded across the first row.

2b) The $(2, 1)$ entry of $(A)(Adj(A))$ is $0\cdot5 + 3\cdot2 + (1)(-6) = 0$. Notice that you're using the same cofactors as in (2a). What matrix, which is obtained from A by keeping two of the rows, but replacing a third one with one of those two, is this the determinant of?

$$row_2(A)\cdot row_1(cof(A)) = \begin{vmatrix} 0 & 3 & 1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{vmatrix} \begin{matrix} \leftarrow row_2(A) \\ \left.\begin{matrix} \\ \\ \end{matrix}\right\} \text{unchanged.} \end{matrix}$$

2c) The $(3, 2)$ entry of $(A)(Adj(A))$ is $2\cdot0 - 2\cdot3 + 1\cdot6 = 0$. What matrix (which uses two rows of A) is this the determinant of?

$$\begin{vmatrix} 1 & 2 & -1 \\ 2 & -2 & 1 \\ 2 & -2 & 1 \end{vmatrix} = 0$$

If you completely understand 2abc, then you have realized why

$$[A][Adj(A)] = det(A)[I]$$

for every square matrix, and so also why

$$A^{-1} = \frac{1}{det(A)} Adj(A) .$$

Precisely,

$$entry_{i\,i}\, A(Adj(A)) = row_i(A) \cdot col_i(Adj(A)) = row_i(A) \cdot row_i(cof(A)) = det(A),$$

expanded across the i^{th} row.

On the other hand, for $i \neq k$,

$$entry_{k\,i}\, A(Adj(A)) = row_k(A) \cdot col_i(Adj(A)) = row_k(A) \cdot row_i(cof(A)) .$$

This last dot produce is zero because it is the determinant of a matrix made from A by replacing the i^{th} row with the k^{th} row, expanding across the i^{th} row, and whenever two rows are equal, the determinant of a matrix is zero:

$$\begin{array}{r} \\ \\ i^{th} \text{ row position} \\ \\ \\ \end{array} \begin{vmatrix} \mathcal{R}_1 \\ \mathcal{R}_2 \\ \mathcal{R}_k \\ \mathcal{R}_k \\ \mathcal{R}_n \end{vmatrix} .$$

There's a related formula for solving for individual components of $\underline{\boldsymbol{x}}$ when $A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}$ has a unique solution ($\underline{\boldsymbol{x}} = A^{-1}\underline{\boldsymbol{b}}$). This can be useful if you only need one or two components of the solution vector, rather than all of it:

<u>Cramer's Rule</u>: Let $\underline{\boldsymbol{x}}$ solve $A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}$, for invertible A. Then

$$x_k = \frac{det(A_k)}{det(A)} .$$

where A_k is the matrix obtained from A by replacing the k^{th} column with $\underline{\boldsymbol{b}}$.

proof: Since $\underline{\boldsymbol{x}} = A^{-1}\underline{\boldsymbol{b}}$ the k^{th} component is given by

$$
\begin{aligned}
x_k &= entry_k\left(A^{-1}\underline{\boldsymbol{b}}\right)\\
&= entry_k\left(\frac{1}{|A|}Adj(A)\underline{\boldsymbol{b}}\right)\\
&= \frac{1}{|A|}row_k(Adj(A)) \cdot \underline{\boldsymbol{b}}\\
&= \frac{1}{|A|}col_k(cof(A)) \cdot \underline{\boldsymbol{b}} .
\end{aligned}
$$

Notice that $col_k(cof(A)) \cdot \underline{\boldsymbol{b}}$ is the determinant of the matrix obtained from A by replacing the k^{th} column by $\underline{\boldsymbol{b}}$, where we've computed that determinant by expanding down the k^{th} column! This proves the result. (See our text for another way of justifying Cramer's rule.)

<u>Exercise 3)</u> Solve $\begin{bmatrix} 5 & -1 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$.

<u>3a)</u> With Cramer's rule

<u>3b)</u> With A^{-1}, using the adjoint formula. (skip).

$$x = \frac{\begin{vmatrix} \overset{\downarrow b}{7} & -1 \\ 2 & 1 \end{vmatrix}}{\begin{vmatrix} 5 & -1 \\ 4 & 1 \end{vmatrix}} = \frac{9}{9} = 1$$

$$y = \frac{\begin{vmatrix} 5 & \overset{\swarrow b}{7} \\ 4 & 2 \end{vmatrix}}{\begin{vmatrix} 5 & -1 \\ 4 & 1 \end{vmatrix}} = \frac{-18}{9} = -2.$$

$$\begin{bmatrix} 5 & -1 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \end{bmatrix} \checkmark$$

Friday Food for Thought 5

Due Tuesday February 20

Spend the rest of today's class period working through these problems. I encourage you to work with your classmates and discuss the problems. If you are finished with the assignment at the end of class today, then you can turn it in today. If you would like to work on the assignment more, take it home over the weekend and turn it in on Tuesday. This assignment will be graded for ***effort*** *(which means that you have written down thoughtful, complete solutions to each problem), not correctness. Solutions to these problems will be posted on Canvas Tuesday for future reference.*

Let's explore what determinants have to do with linear transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$ (generalizes to the case of linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^n$), and with *affine transformations*, which are compositions of translations and linear transformations. So for today, we'll be thinking about functions of the form

$$F\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix}$$

which transform $\mathbb{R}^2$ to $\mathbb{R}^2$. Since

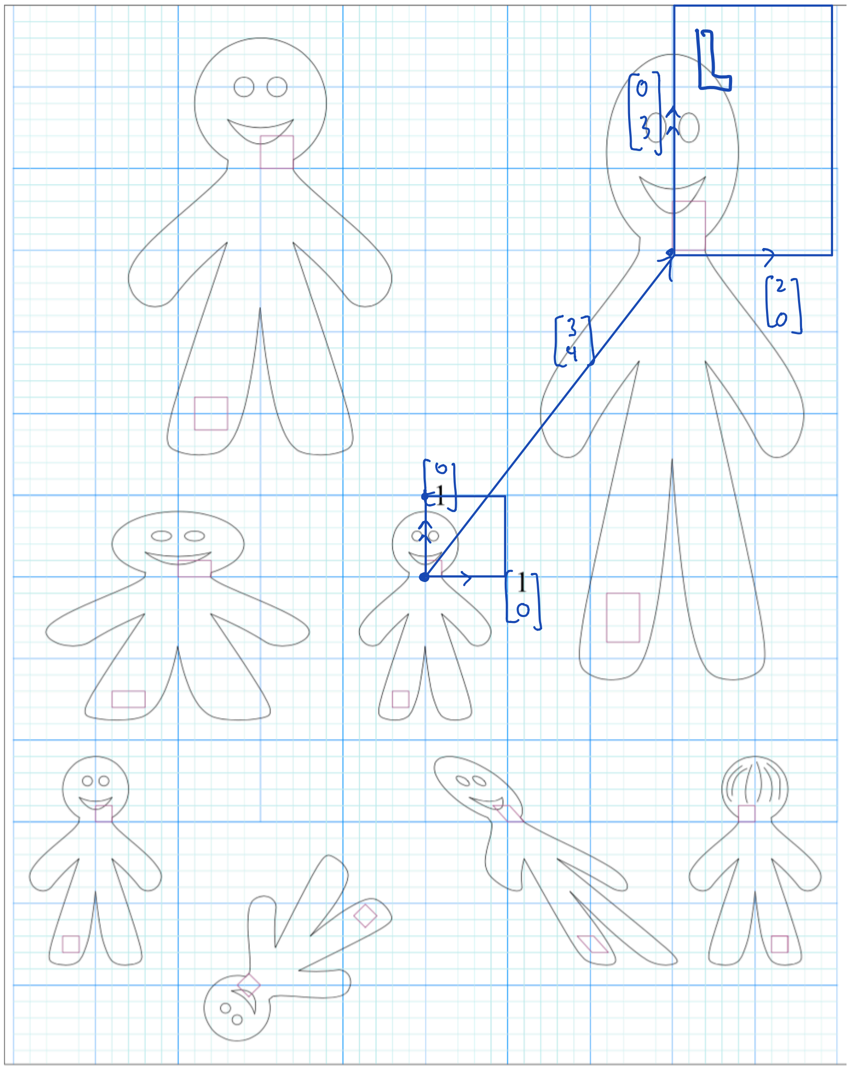
$$F\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} e \\ f \end{bmatrix}$$

$$F\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix}$$

$$F\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} c \\ d \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix},$$

You can reconstruct the formula for the affine function as soon as you know the images of $\underline{\mathbf{0}}$, $\underline{e}_1$, $\underline{e}_2$. For example, I reconstructed the transformation formula for Giant Bob in the upper right corner of the next page. Notice that Giant Bob has six times the area of original Bob - since original Bob can be filled up with different-sized squares, and the images of those squares will be rectangles having six times the original areas.

1) Reconstruct the formulas for at least three more of the six (non-identity) transformations of Bob on the next page, and comment on how the areas of the transformed Bobs are related to the determinants of the matrices in the transformations. Note that the Bob in the lower right corner got flipped over.



$$G\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = \begin{bmatrix}2 & 0\\0 & 3\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} + \begin{bmatrix}3\\4\end{bmatrix}$$

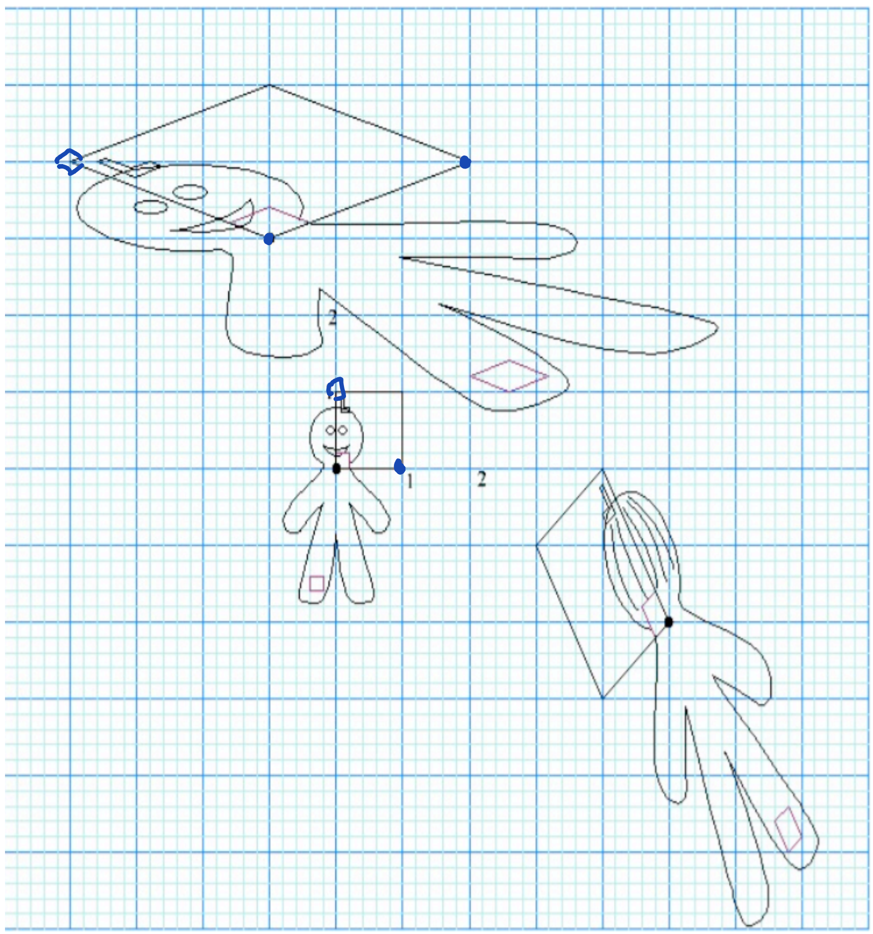
$$F\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix}$$

2) Find the formulas for these two affine transformations of Bob.

$$F\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} e \\ f \end{bmatrix} \quad 1^{st}$$

$$F\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix} \quad \text{for } \begin{bmatrix} a \\ b \end{bmatrix}$$

$$F\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} c \\ d \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix} \quad \text{for } \begin{bmatrix} c \\ d \end{bmatrix}$$



3) Squares in original Bob get transformed into parallelegrams in the image Bobs, and the area expansion factors are independent of the size of the original squares. So, you can deduce the area expansion factor for the image Bobs just by computing the area of the parallegram image of the unit square. How do your area expansion factors in these two examples compare to the matrix determinants from the affine transformations?

We'll talk more systematically about area/volume expansion factors, and in arbitrary dimension on Tuesday, but for affine transformations from $\mathbb{R}^2 \to \mathbb{R}^2$ one can use geometry to connect determinants to area expansion factors.

4) Can you compute the area of the parallelgram below (in terms of the letters a, b, c, d)? Since translations don't effect area, this will give the area expansion factor also for the images of arbitrary regions, under affine transformations

$$F\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix}.$$

Hint: Start with the area of the large rectangle of length $a + c$ and height $b + d$, then subtract off the areas of the triangles and rectangles on the outside of the parallelgram. For convenience I chose the case where all of a, b, c, d are positive, and where the transformation didn't "flip" the parallelgram: