Wed Feb 14

- 3.2 properties of determinants

Announcements:
- quiz today
- fft Friday: likely about geometry of determinants

Warm-up Exercise: compute this determinant:

'til 10:57

$$\begin{matrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{matrix}$$

$$\begin{vmatrix} 1 & 0 & 1 & 2 \\ 2 & 0 & -4 & 2 \\ 7 & 3 & 8 & 6 \\ 0 & 0 & -3 & 0 \end{vmatrix}$$

$|A|$

across any row i:

$$\sum_{j=1}^{n} a_{ij} C_{ij}$$

$$C_{ij} = (-1)^{i+j} M_{ij}$$

or down column j:

$$\sum_{i=1}^{n} a_{ij} C_{ij}$$

col_2 or row_4 are expansion choices.

col_2: $(-0)\cdot M_{12}$

$+0 M_{22}$

$-3 M_{32} + 0 M_{42}$.

$$= -3 \begin{vmatrix} 1 & 1 & 2 \\ 2 & -4 & 2 \\ 0 & -3 & 0 \end{vmatrix}$$

$$= -3\left(0 - (-3)\begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix} + 0\cdot\right)$$

$$= -9(2-4) = 18$$

The effective way to compute determinants for larger-sized matrices without lots of zeroes is to not use the definition, but rather to use the following facts, which track how elementary row operations affect determinants:

- (1a) Swapping any two rows changes the sign of the determinant.

  proof: This is clear for 2 × 2 matrices, since

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc, \qquad \begin{vmatrix} c & d \\ a & b \end{vmatrix} = cb - ad\ .$$

  For 3 × 3 determinants, expand across the row *not* being swapped, and use the 2 × 2 swap property to deduce the result. Prove the general result by induction: once it's true for $n \times n$ matrices you can prove it for any $(n+1) \times (n+1)$ matrix, by expanding across a row that wasn't swapped, and applying the $n \times n$ result.

unswapped:

eg. swap row_1 & row_2

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = +a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} - a_{32}\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} + a_{33}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}.$$

what happens on the right if I swap row1 & row2 on the left?

Ans: rows swap on RHS, so get opposite values from before.

so entire 3×3 determinant changes sign.

so by induction, can show if statement is true for all n×n matrices, it's also true for (n+1)×(n+1) matrices.

(1b) Thus, if two rows in a matrix are the same, the determinant of the matrix must be zero: on the one hand, swapping those two rows leaves the matrix and its determinant unchanged; on the other hand, by (1a) the determinant changes its sign. The only way this is possible is if the determinant is zero.

• (2a) If you factor a constant out of a row, then you factor the same constant out of the determinant. Precisely, using $\mathcal{R}_i$ for i^{th} row of A, and writing $\mathcal{R}_i = c\,\mathcal{R}_i^*$

$$\begin{vmatrix} \mathcal{R}_1 \\ \mathcal{R}_2 \\ \vdots \\ \mathcal{R}_i \\ \mathcal{R}_n \end{vmatrix} = \begin{vmatrix} \mathcal{R}_1 \\ \mathcal{R}_2 \\ \vdots \\ c\,\mathcal{R}_i^* \\ \mathcal{R}_n \end{vmatrix} = c \begin{vmatrix} \mathcal{R}_1 \\ \mathcal{R}_2 \\ \vdots \\ \mathcal{R}_i^* \\ \mathcal{R}_n \end{vmatrix}.$$

i^{th} row $\longrightarrow$

$[a_{i1}\ a_{i2}\ \cdots\ a_{in}]$

$= [c a_{i1}^*\ c a_{i2}^*\ \cdots\ c a_{in}^*]$

proof: expand across the i^{th} row, noting that the corresponding cofactors don't change, since they're computed by deleting the i^{th} row to get the corresponding minors:

$$det(A) = \sum_{j=1}^{n} a_{ij} C_{ij} = \sum_{j=1}^{n} c\, a_{ij}^* C_{ij} = c \sum_{j=1}^{n} a_{ij}^* C_{ij} = c\, det(A^*)\,.$$

expand across i^{th} row.

$$\begin{vmatrix} 2 & 4 \\ 9 & 3 \end{vmatrix} = 2 \begin{vmatrix} 1 & 2 \\ 9 & 3 \end{vmatrix} = 2\cdot 3 \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = 2\cdot 3\,(1-6) = 2\cdot 3(-5) = -30$$

$\parallel$

$6 - 36 = -30$

(2b) Combining (2a) with (1b), we see that if one row in A is a scalar multiple of another, then $det(A) = 0$.

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 0 & 8 \\ 1 & 0 & 2 \end{vmatrix} = 4 \begin{vmatrix} 1 & 2 & 3 \\ 1 & 0 & 2 \\ 1 & 0 & 2 \end{vmatrix} = 0 \text{ by } 1b$$

- (3) If you replace row i of A, $\mathcal{R}_i$ by its sum with a multiple of another row, say $\mathcal{R}_k$ then the determinant is unchanged! Expand across the i^{th} row:

$$\begin{array}{r} \\ \\ \\ i^{th}\text{ row location} \\ \\ \end{array}\begin{vmatrix} \mathcal{R}_1 \\ \mathcal{R}_2 \\ \mathcal{R}_k \\ \mathcal{R}_i + c\,\mathcal{R}_k \\ \mathcal{R}_n \end{vmatrix} = \sum_{j=1}^{n} \underbrace{\left(a_{ij} + c\,a_{kj}\right)}_{\text{entries in } i^{th} \text{ row of matrix on left}} C_{ij} = \sum_{j=1}^{n} a_{ij} C_{ij} + c \sum_{j=1}^{n} a_{kj} C_{ij} = det(A) + c \begin{vmatrix} \mathcal{R}_1 \\ \mathcal{R}_2 \\ \mathcal{R}_k \\ \mathcal{R}_k \\ \mathcal{R}_n \end{vmatrix} = det(A) + 0 .$$

Remark: The analogous properties hold for corresponding "elementary column operations". In fact, the proofs are almost identical, except you use column expansions.

Exercise 1) Recompute $\begin{vmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{vmatrix}$ from yesterday (using row and column expansions we always got an answer of 15 then.) This time use elementary row operations (and/or elementary column operations).

$$\begin{vmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{vmatrix} \underset{-2R_1+R_3 \to R_3}{=} \begin{vmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 0 & -6 & 3 \end{vmatrix} \underset{2R_2+R_3\to R_3}{=} \begin{vmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & 5 \end{vmatrix} = 1\cdot 3\cdot 5$$

upper Δ'ular matrix

$$= 5\begin{vmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 15$$

~~= factor 3 out of row 3~~ $\cancel{= 3\begin{vmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 0 & -2 & 1 \end{vmatrix}}$

expand down 1st column.

$$3\left(1\cdot\begin{vmatrix} 3 & 1 \\ -2 & 1 \end{vmatrix} - 0 + 0\right)$$

$$= 3(3+2) = 15 \checkmark$$

Exercise 2) Compute $\begin{vmatrix} 1 & 0 & -1 & 2 \\ 2 & 1 & 1 & 0 \\ 2 & 0 & 1 & 1 \\ -1 & 0 & -2 & 1 \end{vmatrix}$.

Theorem: Let $A_{n \times n}$. Then A^{-1} exists if and only if $det(A) \neq 0$.

proof: We already know that A^{-1} exists if and only if the reduced row echelon form of A is the identity matrix. Now, consider reducing A to its reduced row echelon form, and keep track of how the determinants of the corresponding matrices change: As we do elementary row operations,

- if we swap rows, the sign of the determinant switches.
- if we factor non-zero factors out of rows, we factor the same factors out of the determinants.
- if we replace a row by its sum with a multiple of another row, the determinant is unchanged.

Thus,

$$|A| = c_1|A_1| = c_1c_2|A_2| = \ldots = c_1c_2 \ldots c_N|rref(A)|$$

where the nonzero c_k's arise from the three types of elementary row operations. If $rref(A) = I$ its determinant is 1, and $|A| = c_1c_2 \ldots c_N \neq 0$. If $rref(A) \neq I$ then its bottom row is all zeroes and its determinant is zero, so $|A| = c_1c_2 \ldots c_N(0) = 0$. Thus $|A| \neq 0$ if and only if $rref(A) = I$ if and only if A^{-1} exists !

Theorem: Using the same ideas as above, we can show that $det(AB) = det(A)det(B)$. This is an important identity that gets used, for example, in multivariable change of variables formulas for integration, using the Jacobian matrix. (It is not true that $det(A+B) = det(A) + det(B)$.)

Here's how to show $det(AB) = det(A)det(B)$: The key point is that if you do an elementary row operation to AB, that's the same as doing the elementary row operation to A, and then multiplying by B. With that in mind, if you do exactly the same elementary row operations as you did for A in the theorem above, you get

$$|AB| = c_1|A_1B| = c_1c_2|A_2B| = \ldots = c_1c_2 \ldots c_N|rref(A)B|.$$

If $rref(A) = I$, then from the theorem above, $|A| = c_1c_2 \ldots c_N$, and we deduce $|AB| = |A||B|$. If $rref(A) \neq I$, then its bottom row is zeroes, and so is the bottom row of $rref(A)B$. Thus $|AB| = 0$ and also $|A||B| = 0$.