Wed Feb 14

- 3.2 properties of determinants

Announcements:
- Quiz today
- FFT Friday: likely about geometry of determinants

Warm-up Exercise:

Compute this determinant:
\[
\begin{vmatrix}
1 & 0 & 1 & 2 \\
2 & 0 & 4 & 2 \\
7 & 3 & 8 & 6 \\
0 & 0 & -3 & 0
\end{vmatrix}
\]

\[
\text{col}_2 \text{ or row}_4 \\
\text{one expansion choices.}
\]

\[
\begin{align*}
\text{col}_2: & (-0) \cdot M_{12} + 0 \cdot M_{22} - 3 \cdot M_{32} + 0 \cdot M_{42}. \\
&= -3 \begin{vmatrix}
1 & 1 & 2 \\
2 & -4 & 2 \\
0 & -3 & 0
\end{vmatrix}
\]
\end{align*}
\]

\[
= -3 \left( 0 - (-3) \begin{vmatrix} 1 \ 2 \\ 2 \ 2 \end{vmatrix} + 0 \cdot 0 \right) \\
= -9 \left( 2 - 4 \right) = 18
\]
The effective way to compute determinants for larger-sized matrices without lots of zeroes is to not use the
definition, but rather to use the following facts, which track how elementary row operations affect
determinants:

- (1a) Swapping any two rows changes the sign of the determinant.
  
  *Proof:* This is clear for $2 \times 2$ matrices, since

  \[
  \begin{vmatrix}
  a & b \\
  c & d \\
  \end{vmatrix} = ad - bc,
  \quad
  \begin{vmatrix}
  c & d \\
  a & b \\
  \end{vmatrix} = cb - ad.
  \]

  For $3 \times 3$ determinants, expand across the row not being swapped, and use
  the $2 \times 2$ swap property to deduce the result. Prove the general result by induction:
  once it's true for $n \times n$ matrices you can prove it for any $(n+1) \times (n+1)$ matrix,
  by expanding across a row that wasn't swapped, and applying the $n \times n$ result.

- (1b) Thus, if two rows in a matrix are the same, the determinant of the matrix must be zero:
  on the one hand, swapping those two rows leaves the matrix and its determinant unchanged;
  on the other hand, by (1a) the determinant changes its sign. The only way this is possible
  is if the determinant is zero.
• (2a) If you factor a constant out of a row, then you factor the same constant out of the determinant.

Precisely, using \( R_i \) for \( i^{th} \) row of \( A \), and writing \( R_i = c \cdot R_i^* \)

\[
\begin{bmatrix}
R_1 \\
R_2 \\
\vdots \\
R_i \\
R_n
\end{bmatrix} =
\begin{bmatrix}
R_1 \\
R_2 \\
\vdots \\
R_i \\
R_n
\end{bmatrix} =
\begin{bmatrix}
R_1 \\
R_2 \\
\vdots \\
R_i \\
R_n
\end{bmatrix} 
\]

**proof:** expand across the \( i^{th} \) row, noting that the corresponding cofactors don't change, since they're computed by deleting the \( i^{th} \) row to get the corresponding minors:

\[
\det(A) = \sum_{j=1}^{n} a_{ij} C_{ij} = \sum_{j=1}^{n} c_{ij}^* C_{ij} = c \sum_{j=1}^{n} a_{ij} C_{ij} = c \det(A^*).
\]

\[
\begin{vmatrix}
2 & 4 \\
9 & 3
\end{vmatrix} =
\begin{vmatrix}
1 & 2 \\
9 & 3
\end{vmatrix} =
\begin{vmatrix}
1 & 2 \\
3 & 1
\end{vmatrix} =
2 \cdot 3 \cdot (-6) = 2 \cdot 3 \cdot (-5) = -30
\]

(2b) Combining (2a) with (1b), we see that if one row in \( A \) is a scalar multiple of another, then \( \det(A) = 0 \)

\[
\begin{vmatrix}
1 & 2 & 3 \\
4 & 0 & 8 \\
1 & 0 & 2
\end{vmatrix} = 4 \begin{vmatrix}
1 & 2 & 3 \\
1 & 0 & 2 \\
1 & 0 & 2
\end{vmatrix} = 0 \text{ by (1b)}
\]
(3) If you replace row $i$ of $A$, $\mathcal{R}_i$, by its sum with a multiple of another row, say $\mathcal{R}_k$, then the determinant is unchanged! Expand across the $i^{th}$ row:

$$
\mathcal{R}_1 \\
\mathcal{R}_2 \\
\mathcal{R}_k \\
\mathcal{R}_i + c \mathcal{R}_k \\
\mathcal{R}_n
$$

$$
\begin{align*}
\left| \begin{array}{c}
\mathcal{R}_1 \\
\mathcal{R}_2 \\
\mathcal{R}_k \\
\mathcal{R}_i + c \mathcal{R}_k \\
\mathcal{R}_n
\end{array} \right| &= \sum_{j=1}^{n} (a_{ij} + c a_{kj}) C_{ij} = \sum_{j=1}^{n} a_{ij} C_{ij} + c \sum_{j=1}^{n} a_{kj} C_{ij} = \det(A) + cC_{ij} = \det(A) + 0 .
\end{align*}
$$

**Remark:** The analogous properties hold for corresponding "elementary column operations". In fact, the proofs are almost identical, except you use column expansions.
Exercise 1) Recompute \[
\begin{bmatrix}
1 & 2 & -1 \\
0 & 3 & 1 \\
2 & -2 & 1 \\
\end{bmatrix}
\]
from yesterday (using row and column expansions we always got an answer of 15 then.) This time use elementary row operations (and/or elementary column operations).

Exercise 2) Compute
\[
\begin{bmatrix}
1 & 0 & -1 & 2 \\
2 & 1 & 1 & 0 \\
2 & 0 & 1 & 1 \\
-1 & 0 & -2 & 1 \\
\end{bmatrix}
\]
Theorem: Let \( A_{n \times n} \). Then \( A^{-1} \) exists if and only if \( \text{det}(A) \neq 0 \).

proof: We already know that \( A^{-1} \) exists if and only if the reduced row echelon form of \( A \) is the identity matrix. Now, consider reducing \( A \) to its reduced row echelon form, and keep track of how the determinants of the corresponding matrices change: As we do elementary row operations,

- if we swap rows, the sign of the determinant switches.
- if we factor non-zero factors out of rows, we factor the same factors out of the determinants.
- if we replace a row by its sum with a multiple of another row, the determinant is unchanged.

Thus,

\[
|A| = c_1|A_1| = c_1c_2|A_2| = \ldots = c_1c_2 \ldots c_N |\text{rref}(A)|
\]

where the nonzero \( c_k \)'s arise from the three types of elementary row operations. If \( \text{rref}(A) = I \) its determinant is 1, and \( |A| = c_1c_2 \ldots c_N \neq 0 \). If \( \text{rref}(A) \neq I \) then its bottom row is all zeroes and its determinant is zero, so \( |A| = c_1c_2 \ldots c_N(0) = 0 \). Thus \( |A| \neq 0 \) if and only if \( \text{rref}(A) = I \) if and only if \( A^{-1} \) exists!

Theorem: Using the same ideas as above, we can show that \( \text{det}(AB) = \text{det}(A)\text{det}(B) \). This is an important identity that gets used, for example, in multivariable change of variables formulas for integration, using the Jacobian matrix. (It is not true that \( \text{det}(A + B) = \text{det}(A) + \text{det}(B) \).)

Here's how to show \( \text{det}(AB) = \text{det}(A)\text{det}(B) \): The key point is that if you do an elementary row operation to \( AB \), that's the same as doing the elementary row operation to \( A \), and then multiplying by \( B \). With that in mind, if you do exactly the same elementary row operations as you did for \( A \) in the theorem above, you get

\[
|AB| = c_1|A_1B| = c_1c_2|A_2B| = \ldots = c_1c_2 \ldots c_N |\text{rref}(A)B|.
\]

If \( \text{rref}(A) = I \), then from the theorem above, \( |A| = c_1c_2 \ldots c_N \), and we deduce \( |AB| = |A||B| \). If \( \text{rref}(A) \neq I \), then its bottom row is zeroes, and so is the bottom row of \( \text{rref}(A)B \). Thus \( |AB| = 0 \) and also \( |A||B| = 0 \).