Tues Feb 13

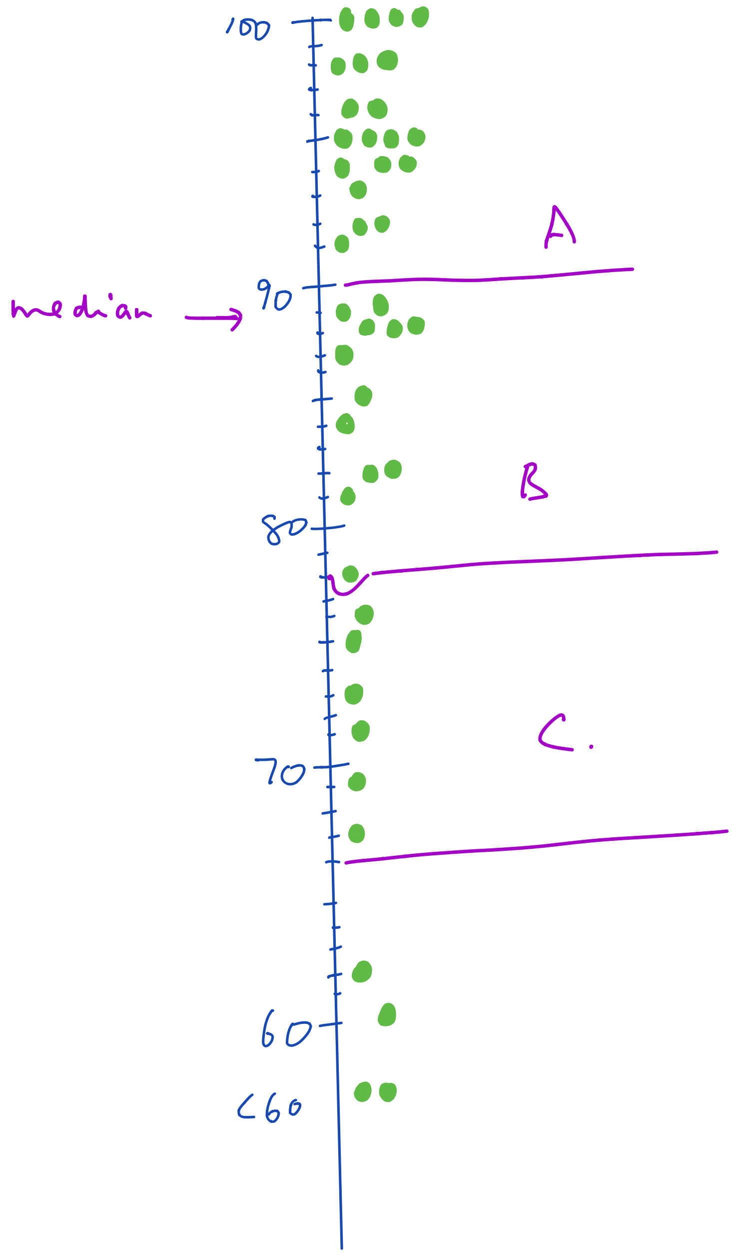
- 3.1 determinants

Announcements: look over your exams. Class did great! • a few struggled.

quiz tomorrow on what we cover today & tomorrow

Warm-up Exercise: new topic – no warmup. ☺

Exam 1
2270-004
n=42



Determinants are scalars defined for square matrices $A_{n \times n}$ They always determine whether or not the inverse matrix A^{-1} exists, (i.e. whether the reduced row echelon form of A is the identity matrix): In fact, the determinant of A is non-zero if and only if A^{-1} exists. The determinant of a 1×1 matrix $[a_{11}]$ is defined to be the number a_{11}. (And whether or not $a_{11} = 0$ determines if it doesn't or does have a multiplicative inverese.) Determinants of 2×2 matrices are defined as in or magic formula for inverse matrices, in the 2×2 case; and in general determinants for $n \times n$ matrices are defined recursively, in terms of determinants of $(n - 1) \times (n - 1)$ submatrices:

Definition: Let $A_{n \times n} = [a_{ij}]$. Then the determinant of A, written $det(A)$ or $|A|$, is defined by

$$det(A) := \sum_{j=1}^{n} a_{1j}(-1)^{1+j}M_{1j} = \sum_{j=1}^{n} a_{1j}C_{1j}.$$

$(\pm)$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ \vdots & & & \end{bmatrix}$$

M_{11} is det of submatrix

Here M_{1j} is the determinant of the $(n - 1) \times (n - 1)$ matrix obtained from A by deleting the first row and the j^{th} column, and $\boxed{C_{1j} \text{ is simply } (-1)^{1+j}M_{1j}.}$

More generally, the determinant of the $(n - 1) \times (n - 1)$ matrix obtained by deleting row i and column j from A is called the ij Minor M_{ij} of A, and $C_{ij} := (-1)^{i+j}M_{ij}$ is called the ij Cofactor of A.

Exercise 1 Check that the messy looking definition above gives the same answer we talked about in regards to our formula for the inverse of 2×2 matrices, namely

"absolute value - like vertical lines, are saying determinant

know:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}.$$

- use inductive def: $\sum_{j=1}^{2} a_{1j}(-1)^{1+j} M_{1j}$

$$= a_{11} \cdot 1 \cdot a_{22}$$

$$+ a_{12}(-1)a_{21}$$

$$= a_{11}a_{22} - a_{12}a_{21}$$

$$\begin{matrix} \cancel{a_{11}} & \cancel{a_{12}} \\ a_{21} & a_{22} \end{matrix} \qquad \begin{matrix} \cancel{a_{11}} & \cancel{a_{12}} \\ a_{21} & a_{22} \end{matrix}$$

✓

from the last page, for our convenience:

Definition: Let $A_{n \times n} = [a_{ij}]$. Then the determinant of A, written $det(A)$ or $|A|$, is defined by

$$\boxed{det(A) := \sum_{j=1}^{n} a_{1j}(-1)^{1+j}M_{1j} = \sum_{j=1}^{n} a_{1j}C_{1j}.} \qquad n=3.$$

Here M_{1j} is the determinant of the $(n-1) \times (n-1)$ matrix obtained from A by deleting the first row and the j^{th} column, and C_{1j} is simply $(-1)^{1+j}M_{1j}$.

Exercise 2 Work out the expanded formula for the determinant of a 3×3 matrix. It's not worth memorizing (as opposed to the recursive formula above), but it's good practice to write out at least once, and we might point to it later.

$$\left[(-1)^{i+j}\right] = \begin{bmatrix} + & - & + & \cdots \\ - & + & - & \\ + & - & + & \\ \vdots & & & \end{bmatrix}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

6 terms

$6 = 3!$

$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

note : each triple product use each row & column exactly once

if order terms so that rows go 1, 2, 3

then $\pm$ is the sign of the permutation of 1, 2, 3 that gives the columns

see wikipedia

Exercise 3a) Let $A := \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{bmatrix}$. Compute $det(A)$ using the definition. (On the next page we'll use other rows and columns to do the computation.)

$$|A| = 1 \cdot \begin{vmatrix} 3 & 1 \\ -2 & 1 \end{vmatrix} + 2\left(-\begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix}\right) + (-1)\begin{vmatrix} 0 & 3 \\ 2 & -2 \end{vmatrix}$$

$$= 1 \cdot (3 - -2) + 2(-1)(0-2) + (-1)(0-6)$$

$$= 5 + 4 + 6 = 15 \;!$$

Theorem: $det(A)$ can be computed by expanding across any row, say row i:

$$det(A) := \sum_{j=1}^{n} a_{ij}(-1)^{i+j}M_{ij} = \sum_{j=1}^{n} a_{ij}C_{ij} \qquad row_i(A)$$

or by expanding down any column, say column j:

$$det(A) := \sum_{i=1}^{n} a_{ij}(-1)^{i+j}M_{ij} = \sum_{i=1}^{n} \underset{\text{"factor"}}{a_{ij}}\,\underset{\text{"cofactor"}}{C_{ij}}. \qquad col_j(A)$$

(proof is not so easy - our text skips it and so will we. If you look on Wikipedia you'll see that the determinant is actually a sum of n factorial terms, each of which is $\pm$ a product of n entries of A where each product has exactly one entry from each row and column. The $\pm$ sign has to do with whether the corresponding permutation is even or odd. You can verify this pretty easily for the 2×2 and 3×3 cases. Then one shows inductively that each row or column cofactor expansion reproduces this sum, in the $n \times n$ case.)

From previous page,

$$A := \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{bmatrix}.$$

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix} = \left[(-1)^{i+j}\right]$$

<u>3b)</u> Verify that the matrix of all the cofactors of A is given by $[C_{ij}] = \begin{bmatrix} 5 & 2 & -6 \\ 0 & 3 & 6 \\ 5 & -1 & 3 \end{bmatrix}$. Then expand $det(A)$ down various columns and rows using the a_{ij} factors and C_{ij} cofactors. Verify that you always get the same value for $det(A)$, as the Theorem on the previous page guarantees. Notice that in each case you are taking the dot product of a row (or column) of A with the corresponding row (or column) of the cofactor matrix.

$$A := \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{bmatrix} \qquad [C_{ij}] = \begin{bmatrix} 5 & 2 & -6 \\ 0 & 3 & 6 \\ 5 & -1 & 3 \end{bmatrix}$$

$$[C_{ij}] = \begin{bmatrix} +\begin{vmatrix} 3 & 1 \\ -2 & 1 \end{vmatrix} & -\begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} & +\begin{vmatrix} 0 & 3 \\ 2 & -2 \end{vmatrix} \\ -\begin{vmatrix} 2 & -1 \\ -2 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 2 & -2 \end{vmatrix} \\ +\begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 5 & 2 & -6 \\ 0 & 3 & 6 \\ 5 & -1 & 3 \end{bmatrix} \checkmark$$

did $row_1(A) \cdot row_1(cof(A))$

$row_2(A) \cdot row_2(cof(A)) = 0 + 3\cdot 3 + 1\cdot 6 = 15$,

$col_3(A) \cdot col_3(cof(A)) = 6 + 6 + 3 = 15$

$col_1(A) \cdot col_1(cof(A)) = 5 + 0 + 10 = 15$

3c) What happens if you take dot products between a row of A and a *different* row of $[C_{ij}]$? A column of A and a *different* column of $[C_{ij}]$? The answer may seem magic.

$$A := \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{bmatrix} \qquad [C_{ij}] = \begin{bmatrix} 5 & 2 & -6 \\ 0 & 3 & 6 \\ 5 & -1 & 3 \end{bmatrix}$$

$row_1(A) \cdot row_2(cof(A)) = 0 + 6 - 6 = 0$

$row_2(A) \cdot row_3(cof(A)) = 0 - 3 + 3 = 0$

$col_2(A) \cdot col_3(cof(A)) = -12 + 18 - 6 = 0$

we always get 0!

3d) The adjoint matrix is defined to be the transpose of the cofactor matrix. So in our example,

$$A := \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{bmatrix} \qquad adj(A) = (cof(A))^T = \begin{bmatrix} 5 & 0 & 5 \\ 2 & 3 & -1 \\ -6 & 6 & 3 \end{bmatrix}. \qquad cof(A) = \begin{bmatrix} 5 & 2 & -6 \\ 0 & 3 & 6 \\ 5 & -1 & 3 \end{bmatrix}$$

Reinterpret your work in 3bc to say that

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{bmatrix}\begin{bmatrix} 5 & 0 & 5 \\ 2 & 3 & -1 \\ -6 & 6 & 3 \end{bmatrix} = \begin{bmatrix} 15 & 0 & 0 \\ 0 & 15 & 0 \\ 0 & 0 & 15 \end{bmatrix}.$$

so $A^{-1} = \frac{1}{|A|} adj(A)$!

So, in this case - and in fact always, the magic formula for A^{-1} is given by

$$A^{-1} = \frac{1}{det(A)} adj(A).$$

It seems like magic now, but we'll be able to understand why it's true after we learn about more determinant properties on Wednesday and Friday.

Exercise 4) Compute the following determinants by being clever about which rows or columns to use:

4a) $\begin{vmatrix} 1 & 38 & 106 & 3 \\ 0 & 2 & 92 & -72 \\ 0 & 0 & 3 & 45 \\ 0 & 0 & 0 & -2 \end{vmatrix} = |A|$;

4b) $\begin{vmatrix} 1 & 0 & 0 & 0 \\ \pi^2 & 2 & 0 & 0 \\ 0.476 & 88 & 3 & 0 \\ 1 & 22 & 33 & -2 \end{vmatrix} = |B|$.

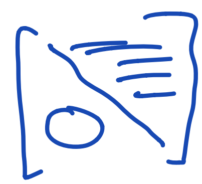
$$\begin{matrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{matrix}$$

4a) 4^{th} row or 1^{st} column.

$$|A| = 1 \begin{vmatrix} 2 & 92 & -72 \\ 0 & 3 & 45 \\ 0 & 0 & -2 \end{vmatrix} - 0 + 0 - 0$$

$$= 1 \cdot 2 \begin{vmatrix} 3 & 45 \\ 0 & -2 \end{vmatrix} - 0 + 0$$

$$= 1 \cdot 2 \cdot 3 \cdot (-2)$$

the matrix 

Exercise 5) Explain why it is always true that for an upper triangular matrix (as in 2a), or for a lower triangular matrix (as in 2b), the determinant is always just the product of the diagonal entries.