on Friday.

Exercise 1

1a) Check that the set

$$B = \left\{ \frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \frac{1}{3}\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, \frac{1}{3}\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} \right\}$$

is an orthonormal basis for $\mathbb{R}^3$.

Checked on Wed warmup.

1b) For $\underline{\boldsymbol{x}} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ find the coordinate vector $[\underline{\boldsymbol{x}}]_B$ and check your answer.

$$\vec{x} = (\vec{x}\cdot\vec{u}_1)\,\vec{u}_1 + (\vec{x}\cdot\vec{u}_2)\,\vec{u}_2 + (\vec{x}\cdot\vec{u}_3)\,u_3$$

$$\underbrace{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\cdot\underbrace{\frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}}_{\vec{u}_1}}_{3} \qquad \underbrace{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\cdot\underbrace{\frac{1}{3}\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}}_{\vec{u}_2}}_{1} \qquad \underbrace{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\cdot\underbrace{\frac{1}{3}\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}}_{\vec{u}_3}}_{2}$$

solution $[\underline{\boldsymbol{x}}]_B = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$ ✓

Check does $\vec{x} = 3\vec{u}_1 + \vec{u}_2 + 2\vec{u}_3 = 3\frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} + 2\frac{1}{3}\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$

$$\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \end{bmatrix} + \begin{bmatrix} -4/3 \\ 2/3 \\ 4/3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \;!$$

Remark: A basis $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \dots, \underline{\boldsymbol{v}}_p\} \subseteq \mathbb{R}^n$ is called *orthogonal* if the the vectors in the set are mutually perpendicular, but not necessarily normalized to unit length. One can construct an orthonormal basis from that set by normalizing, namely

$$\{\underline{\boldsymbol{u}}_1, \underline{\boldsymbol{u}}_2, \dots, \underline{\boldsymbol{u}}_p\} = \left\{ \frac{\underline{\boldsymbol{v}}_1}{\|\underline{\boldsymbol{v}}_1\|}, \frac{\underline{\boldsymbol{v}}_2}{\|\underline{\boldsymbol{v}}_2\|}, \dots, \frac{\underline{\boldsymbol{v}}_p}{\|\underline{\boldsymbol{v}}_p\|} \right\} \subseteq \mathbb{R}^n .$$

One can avoid square roots if one uses the original orthogonal ~~matrix~~ basis rather than the ortho-normal one. This is the approach the text prefers. For example, for orthogonal bases, the very good basis theorem reads

Theorem (why orthogonal bases are very good bases): Let $B = \{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \dots, \underline{\boldsymbol{v}}_p\} \subseteq \mathbb{R}^n$ be orthogonal. Let $W = span\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \dots, \underline{\boldsymbol{v}}_p\}$. Then

a) $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \dots, \underline{\boldsymbol{v}}_p\}$ is linearly independent, so a basis for W.

b) For $\underline{\boldsymbol{w}} \in W$,

$$\underline{\boldsymbol{w}} = (\underline{\boldsymbol{u}}_1 \bullet \underline{\boldsymbol{w}})\underline{\boldsymbol{u}}_1 + (\underline{\boldsymbol{u}}_2 \bullet \underline{\boldsymbol{w}})\underline{\boldsymbol{u}}_2 + \dots + (\underline{\boldsymbol{u}}_p \bullet \underline{\boldsymbol{w}})\underline{\boldsymbol{u}}_p$$

$$\underline{\boldsymbol{w}} = \frac{(\underline{\boldsymbol{v}}_1 \bullet \underline{\boldsymbol{w}})}{\|\underline{\boldsymbol{v}}_1\|^2}\underline{\boldsymbol{v}}_1 + \frac{(\underline{\boldsymbol{v}}_2 \bullet \underline{\boldsymbol{w}})}{\|\underline{\boldsymbol{v}}_2\|^2}\underline{\boldsymbol{v}}_2 + \dots + \frac{(\underline{\boldsymbol{v}}_p \bullet \underline{\boldsymbol{w}})}{\|\underline{\boldsymbol{v}}_p\|^2}\underline{\boldsymbol{v}}_p$$

(no $\sqrt{\ }$'s)

c) Let $\underline{\boldsymbol{x}} \in \mathbb{R}^n$. Then there is a unique nearest point to $\underline{\boldsymbol{x}}$ in W, which we call $proj_W \underline{\boldsymbol{x}}$, ("the projection of $\underline{\boldsymbol{x}}$ onto W.") The formula for this projection is given by

$$proj_W \underline{\boldsymbol{x}} = (\underline{\boldsymbol{u}}_1 \bullet \underline{\boldsymbol{x}})\underline{\boldsymbol{u}}_1 + (\underline{\boldsymbol{u}}_2 \bullet \underline{\boldsymbol{x}})\underline{\boldsymbol{u}}_2 + \dots + (\underline{\boldsymbol{u}}_p \bullet \underline{\boldsymbol{x}})\underline{\boldsymbol{u}}_p .$$

$$proj_W \underline{\boldsymbol{x}} = \frac{(\underline{\boldsymbol{v}}_1 \bullet \underline{\boldsymbol{x}})}{\|\underline{\boldsymbol{v}}_1\|^2}\underline{\boldsymbol{v}}_1 + \frac{(\underline{\boldsymbol{v}}_2 \bullet \underline{\boldsymbol{x}})}{\|\underline{\boldsymbol{v}}_2\|^2}\underline{\boldsymbol{v}}_2 + \dots + \frac{(\underline{\boldsymbol{v}}_p \bullet \underline{\boldsymbol{x}})}{\|\underline{\boldsymbol{v}}_p\|^2}\underline{\boldsymbol{v}}_p .$$

You can see how that would have played out in the previous exercise.

Name Solutions Student I.D.

Math 2270-004 Quiz Week 12 April 4, 2018

1a) Let $L = span\left\{\begin{bmatrix} 3 \\ -1 \end{bmatrix}\right\}$ in $\mathbb{R}^2$. Let $\underline{\boldsymbol{x}} = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$. Compute $proj_L\, \underline{\boldsymbol{x}}$.

no warmup check quiz!

(6 points)

$$proj_L \vec{x} = (\vec{x} \cdot \vec{u})\vec{u}$$

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{1}{\sqrt{10}}\begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$proj_L \vec{x} = \left(\begin{bmatrix} 0 \\ 5 \end{bmatrix} \cdot \frac{1}{\sqrt{10}}\begin{bmatrix} 3 \\ -1 \end{bmatrix}\right)\frac{1}{\sqrt{10}}\begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$= \frac{1}{10}(-5)\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -1.5 \\ .5 \end{bmatrix}$$

$\cancel{\begin{bmatrix} 3 \\ 1 \end{bmatrix}}$

1b) Use the dot product to verify that the vector $\underline{\boldsymbol{z}}$ from $proj_L\, \underline{\boldsymbol{x}}$ to $\underline{\boldsymbol{x}}$ is perpendicular to $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$.

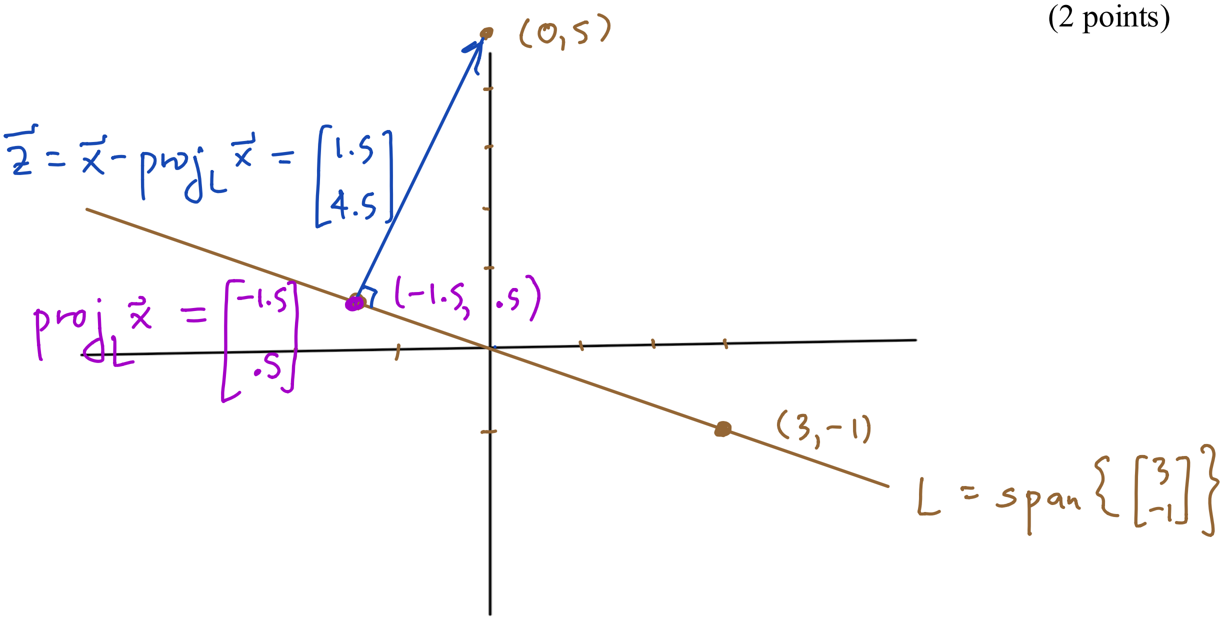
(2 points)

$$\vec{z} = \vec{x} - proj_L \vec{x}$$

$$= \begin{bmatrix} 0 \\ 5 \end{bmatrix} - \begin{bmatrix} -1.5 \\ .5 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 4.5 \end{bmatrix} = 1.5\begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad \cdot \quad 1.5\begin{bmatrix} 1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -1 \end{bmatrix} = 0$$

1c) Make a sketch which illustrates your work in parts a,b. It should include the line L, the points with position vectors $\underline{\boldsymbol{x}}$, $proj_L\, \underline{\boldsymbol{x}}$, and the vector $\underline{\boldsymbol{z}}$ from $proj_L\, \underline{\boldsymbol{x}}$ to $\underline{\boldsymbol{x}}$.

(2 points)



Fri Apr 6

- 6.3-6.4 Gram-Schmidt process for constructing ortho-normal (or orthogonal) bases. The $A = Q\,R$ matrix factorization.

Announcements: 6.4 HW: 1, 3, 5, 7, 11, 13, 15

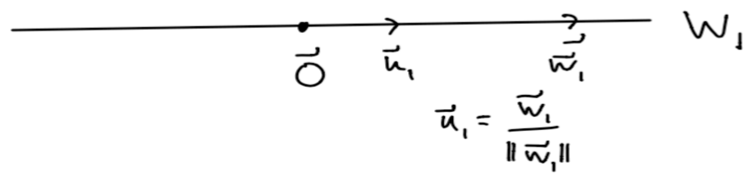
I'll get complete assignment posted by Saturday.

Example in Wed notes, then today's.

Warm-up Exercise: no warm-up → check your quiz

Start with a basis $B = \{\underline{\boldsymbol{w}}_1, \underline{\boldsymbol{w}}_2, \dots \underline{\boldsymbol{w}}_p\}$ for a subspace W of $\mathbb{R}^n$. How can you convert it into an orthonormal basis? Here's how! The inductive process is called Gram-Schmidt orthogonalization.

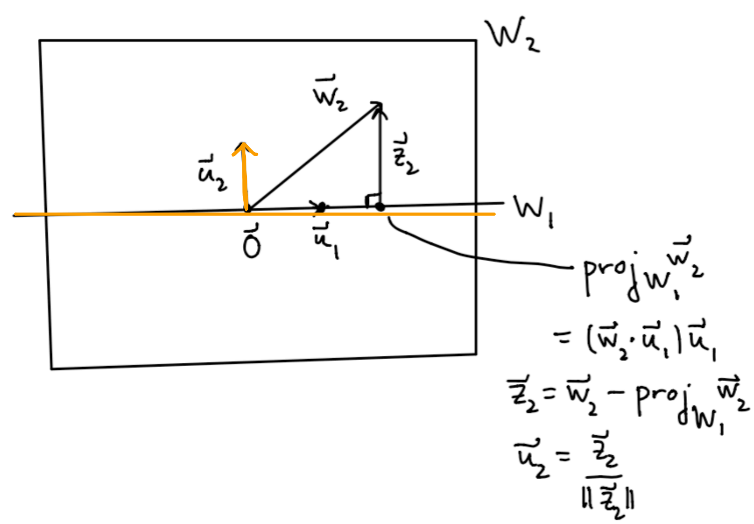
Let $W_1 = span\{\underline{\boldsymbol{w}}_1\}$. Define $\underline{\boldsymbol{u}}_1 = \dfrac{\underline{\boldsymbol{w}}_1}{\|\underline{\boldsymbol{w}}_1\|}$. Then $\{\underline{\boldsymbol{u}}_1\}$ is an orthonormal basis for W_1.



Let $W_2 = span\{\underline{\boldsymbol{w}}_1, \underline{\boldsymbol{w}}_2\}$. $= span\{\vec{u}_1, \vec{w}_2\}$

Let $\underline{\boldsymbol{z}}_2 = \underline{\boldsymbol{w}}_2 - proj_{W_1}\underline{\boldsymbol{w}}_2$, so $\underline{\boldsymbol{z}}_2 \perp \underline{\boldsymbol{u}}_1$. $\vec{z}_2 = \vec{w}_2 - (\vec{w}_2 \cdot \vec{u}_1)\vec{u}_1$ $\quad \vec{z}_2 \cdot \vec{u}_1 = \vec{w}_2 \cdot \vec{u}_1 - \vec{w}_2 \cdot \vec{u}_1 = 0$

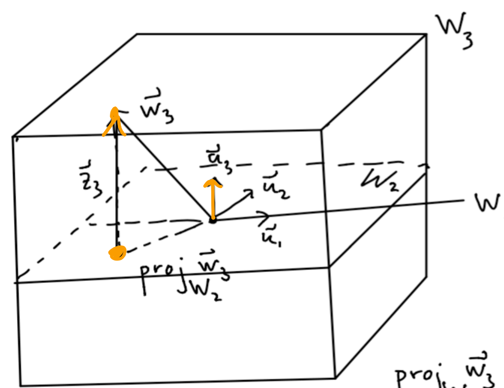
Define $\underline{\boldsymbol{u}}_2 = \dfrac{\underline{\boldsymbol{z}}_2}{\|\underline{\boldsymbol{z}}_2\|}$. So $\{\underline{\boldsymbol{u}}_1, \underline{\boldsymbol{u}}_2\}$ is an orthonormal basis for W_2.



Let $W_3 = span\{\underline{\boldsymbol{w}}_1, \underline{\boldsymbol{w}}_2, \underline{\boldsymbol{w}}_3\}$. $= span\{\vec{u}_1, \vec{u}_2, \vec{w}_3\}$

Let $\underline{\boldsymbol{z}}_3 = \underline{\boldsymbol{w}}_3 - proj_{W_2}\underline{\boldsymbol{w}}_3$, so $\underline{\boldsymbol{z}}_3 \perp W_2$. $\vec{z}_3 = \vec{w}_3 - (\vec{w}_3 \cdot \vec{u}_1)\vec{u}_1 - (\vec{w}_3 \cdot \vec{u}_2)\vec{u}_2$ $\quad \vec{z}_3 \cdot \vec{u}_1 = 0$ $\quad \vec{z}_3 \cdot \vec{u}_2 = 0$

Define $\underline{\boldsymbol{u}}_3 = \dfrac{\underline{\boldsymbol{z}}_3}{\|\underline{\boldsymbol{z}}_3\|}$. Then $\{\underline{\boldsymbol{u}}_1, \underline{\boldsymbol{u}}_2, \underline{\boldsymbol{u}}_3\}$ is an orthonormal basis for W_3.



$proj_{W_2}\vec{w}_3 = (\vec{w}_3 \cdot \vec{u}_1)\vec{u}_1 + (\vec{w}_3 \cdot \vec{u}_2)\vec{u}_2$

$\vec{z}_3 = \vec{w}_3 - proj_{W_2}\vec{w}_3$

$\vec{u}_3 = \dfrac{\vec{z}_3}{\|\vec{z}_3\|}$

Inductively,

Let $W_j = span\{\underline{\boldsymbol{w}}_1, \underline{\boldsymbol{w}}_2, \ldots \underline{\boldsymbol{w}}_j\} = span\{\underline{\boldsymbol{u}}_1, \underline{\boldsymbol{u}}_2, \ldots \underline{\boldsymbol{u}}_{j-1}, \underline{\boldsymbol{w}}_j\}$.

Let $\underline{\boldsymbol{z}}_j = \underline{\boldsymbol{w}}_j - proj_{W_{j-1}}\underline{\boldsymbol{w}}_j = \underline{\boldsymbol{w}}_j - (\underline{\boldsymbol{w}}_j \cdot \underline{\boldsymbol{u}}_1)\underline{\boldsymbol{u}}_1 - (\underline{\boldsymbol{w}}_j \cdot \underline{\boldsymbol{u}}_2)\underline{\boldsymbol{u}}_2 - \cdots (\underline{\boldsymbol{w}}_j \cdot \underline{\boldsymbol{u}}_{j-1})\underline{\boldsymbol{u}}_{j-1}$.

$\vec{z}_j \cdot \vec{u}_1 = 0$
$\vec{z}_j \cdot \vec{u}_2 = 0$
$\vdots$
$\vec{z}_j \cdot \vec{u}_{j-1} = 0$

Define $\underline{\boldsymbol{u}}_j = \dfrac{\underline{\boldsymbol{z}}_j}{\|\underline{\boldsymbol{z}}_j\|}$. Then $\{\underline{\boldsymbol{u}}_1, \underline{\boldsymbol{u}}_2, \ldots \underline{\boldsymbol{u}}_j\}$ is an orthonormal basis for W_j.

Continue up to $j = p$.

Exercise 1 Perform Gram-Schmidt orthogonalization on the basis

$$B = \left\{ \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\vec{w}_1}, \underbrace{\begin{bmatrix} 0 \\ 4 \end{bmatrix}}_{\vec{w}_2} \right\}.$$

Sketch what you're doing, as you do it.

$$\vec{u}_1 = \frac{\vec{w}_1}{\|\vec{w}_1\|} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\vec{z}_2 = \vec{w}_2 - \operatorname{proj}_{W_1} \vec{w}_2$$

$$\vec{z}_2 = \vec{w}_2 - \underbrace{(\vec{w}_2 \cdot \vec{u}_1)\vec{u}_1}$$

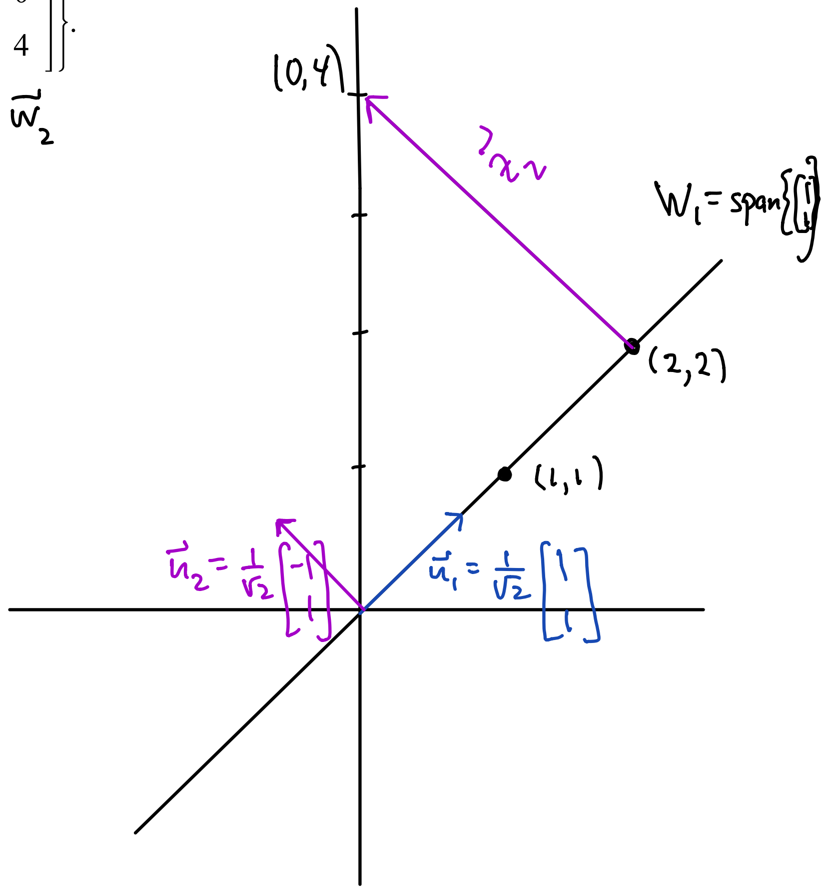
$$\left(\begin{bmatrix} 0 \\ 4 \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\left(\frac{1}{2} \cdot 4 \right) \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$\vec{z}_2 = \begin{bmatrix} 0 \\ 4 \end{bmatrix} - \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix} \quad // \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\vec{u}_2 = \frac{\vec{z}_2}{\|\vec{z}_2\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$



$\vec{w}_1 \quad \vec{w}_2 \quad \vec{w}_3$

Exercise 2 Perform Gram-Schmidt on the basis

$$B = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \right\}.$$

Exercise 1

$$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \end{bmatrix} \right\}$$

This will proceed as in Exercise 1 until the third step, i.e.

$$\underbrace{\boldsymbol{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \boldsymbol{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}_{\text{the spanned the } x_1 - x_2 \text{ plane in } \mathbb{R}^3}, \quad \vec{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{z}_3 = \vec{w}_3 - \text{proj}_{W_2} \vec{w}_3$$

$$\vec{z}_3 = \vec{w}_3 - (\vec{w}_3 \cdot \vec{u}_1)\vec{u}_1 - (\vec{w}_3 \cdot \vec{u}_2)\vec{u}_2$$

$$\vec{z}_3 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} - \left(\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right) \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \left(\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right) \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} - \frac{1}{2}(-1) \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{2}(-3) \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{3}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$$

$$\vec{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$