Theorem (fill in details).

$1 \le p \le n-1$, $0 \le p \le n$

1a) Let $W \subseteq \mathbb{R}^n$ be a subspace with $\dim W = p$, $1 \le p \le n$. Then $\dim(W^\perp) = n - p$, so

$$\dim(W) + \dim(W^\perp) = n$$

Hint: Use reduced row echelon form ideas.

$$\text{Let } A = \underbrace{\begin{bmatrix} \vec{w}_1^T \\ \vec{w}_2^T \\ \vdots \\ \vec{w}_p^T \end{bmatrix}}_{n} \Big\} p$$

where $\{\vec{w}_1, \vec{w}_2, \ldots \vec{w}_p\}$ basis for W.

rref(A) has p pivots

otherwise elementary row ops would've created a basis for W with $< p$ elements, but $\dim W = p$

1b) $W \cap W^\perp = \{\mathbf{0}\}$

$W^\perp = \text{Nul}\, A$, $\dim(\text{Nul}\, A) = n-p$

$=$ # of non-pivot columns.

Hint: Let $\mathbf{x} \in W \cap W^\perp$. Compute $\mathbf{x} \cdot \mathbf{x}$.

intersection

$$\underset{W}{\underset{\cap}{\vec{x}}} \cdot \underset{W^\perp}{\underset{\cap}{\vec{x}}} = 0$$

because $\vec{x} \in W^\perp$, $\vec{x} \cdot \vec{w} = \vec{0}$ $\forall \vec{w} \in W$

1c) $(W^\perp)^\perp = W$.

Hint: Show $W \subseteq (W^\perp)^\perp$. Then count dimensions.

$\vec{u} \in (W^\perp)^\perp$ means $\vec{u} \cdot \vec{z} = 0$ $\forall \vec{z} \in W^\perp$

if $\vec{w} \in W$, and $\vec{z} \in W^\perp$ then $\vec{w} \cdot \vec{z} = 0$

So $\vec{w} \in (W^\perp)^\perp$

$\dim W + \dim W^\perp = n$

$\Rightarrow \dim W^\perp + (\dim W^\perp)^\perp = n$

1d) Let $B = \{\mathbf{w}_1, \mathbf{w}_2, \ldots \mathbf{w}_p\}$ be a basis for W and $C = \{\mathbf{z}_1, \mathbf{z}_2, \ldots \mathbf{z}_{n-p}\}$ be a basis for $W^\perp$. Then their union, $B \cup C$, is a basis for $\mathbb{R}^n$.

$\Rightarrow \dim W = \dim (W^\perp)^\perp$

Hint: Show $B \cup C$ is linearly independent.

but $W \subset (W^\perp)^\perp$

next time!

Since dim's are $=$

$W = (W^\perp)^\perp$

check linear ind.

$c_1\vec{w}_1 + c_2\vec{w}_2 + \ldots + c_p\vec{w}_p + d_1\vec{z}_1 + d_2\vec{z}_2 + \ldots + d_{n-p}\vec{z}_{n-p} = \vec{0}$

$\Rightarrow \underbrace{c_1\vec{w}_1 + c_2\vec{w}_2 + \ldots + c_p\vec{w}_p}_{\in W} = \underbrace{-d_1\vec{z}_1 - d_2\vec{z}_2 - \ldots - d_{n-p}\vec{z}_{n-p}}_{\in W^\perp}$

(the q-dim'l subspace of q-dim'l vectorspace is entire vector space)

1e) Corollary

$W \cap W^\perp = \{\vec{0}\}$

$\Rightarrow c_1\vec{w}_1 + \cdots + c_p\vec{w}_p = 0 \Longrightarrow c_1 = c_2 = \cdots = c_p = 0$

$-d_1\vec{z}_1 - d_2\vec{z}_2 + \cdots - d_{n-p}\vec{z}_{n-p} = 0 \Longrightarrow d_1 = d_2 = \cdots = d_{n-p} = 0$

each $\vec{x} \in \mathbb{R}^n$ can be written uniquely as

$\vec{x} = \vec{w} + \vec{z}$, $\vec{w} \in W$, $\vec{z} \in W^{\perp}$

like proj prob. in $\mathbb{R}^2$

proof: $\vec{x} = \underbrace{c_1\vec{w}_1 + \cdots + c_p\vec{w}_p}_{\vec{w}} + \underbrace{d_1\vec{z}_1 + \cdots + d_{n-p}\vec{z}_{n-p}}_{\vec{z}}$

unique weights

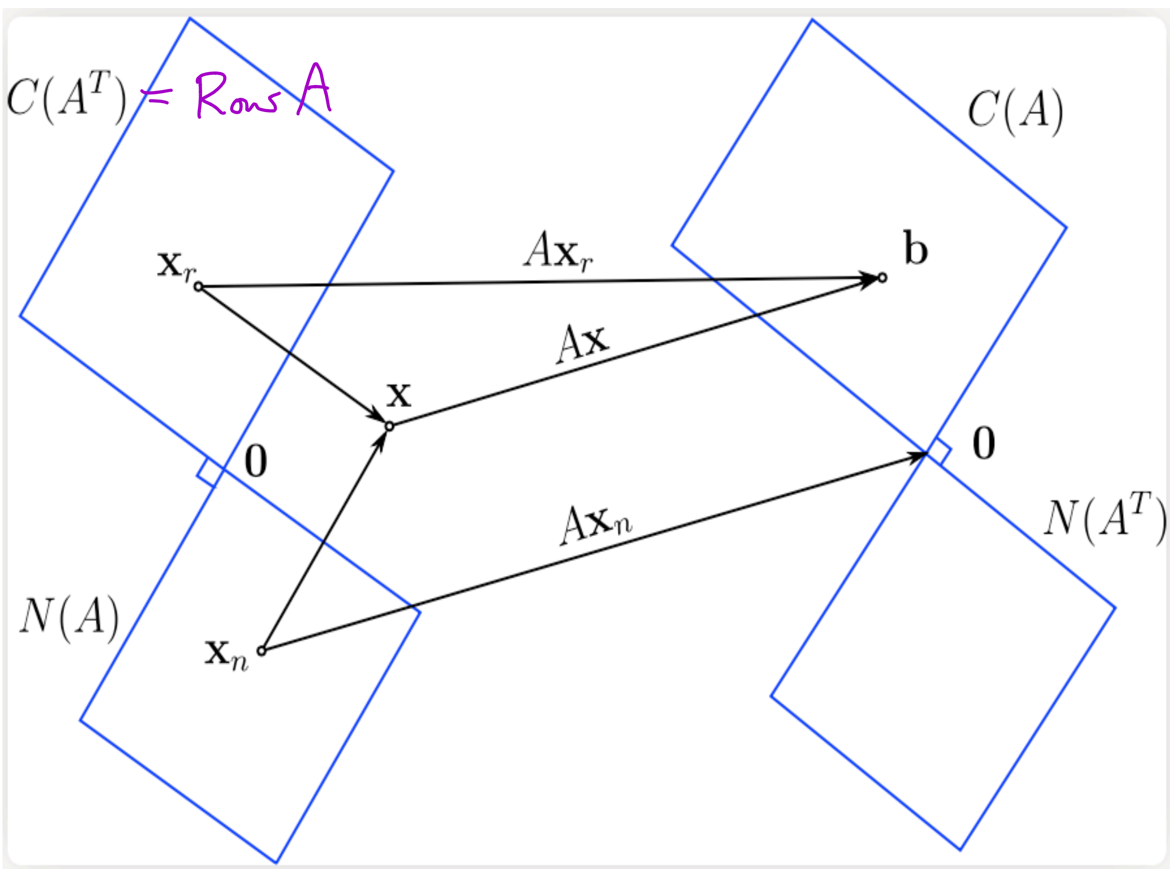
Remark: From the discussion above, and for any $m \times n$ matrix A of arbitrary rank p, we can deduce from the discussion above that $(Row\ A)^{\perp} = Nul\ A$; so $(Nul\ A)^{\perp} = Row\ A$; from our previous work we know that $dim\ (Row\ A) = p$, $dim\,(Nul\ A) = n - p$. This decomposes the domain of the linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$,

rank

$Row\ A^{\perp} = \{\vec{v} \mid \vec{v} \cdot Row_i(A) = 0,\ i = 1, 2, \ldots m\}$
$= Nul\ A$

$$T(\underline{x}) := A\,\underline{x}.$$

By the same reasoning applied to the transpose transformation from $\mathbb{R}^m \to \mathbb{R}^n$, the codomain of T decomposes into $Col\ A = Row\ A^T$ and $(Col\ A)^{\perp} = Nul\ A^T$, with $dim\,(Col\ A) = p$ and $dim\ \left(Nul\ A^T\right) = m - p$. In other words, we have justified the diagram we really only waved our hands at back in Chapter 4, except for transformations from $\mathbb{R}^2 \to \mathbb{R}^2$.



$$\begin{array}{cc} \vec{w}_1 & \vec{w}_2 \end{array}$$

<u>Exercise 2</u>) In <u>Exercise 1</u> with $W = span\left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \right\}$, we showed $W^{\perp} = span\left\{ \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix} \right\}$. Compute $(W^{\perp})^{\perp}$ as $Nul\begin{bmatrix} 2 & -5 & 1 \end{bmatrix}$ and verify that it recovers W (but with a different basis).

$$\begin{bmatrix} 2 & -5 & 1 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = 0$$

$$\begin{array}{ccc|c} 2 & -5 & 1 & 0 \\ \hline 1 & -2.5 & .5 & 0 \end{array}$$

$z_1 = 2.5t_2 - .5t_3$
$z_2 = t_2$
$z_3 = t_3$

$$\begin{array}{ccc} & \vec{v}_1 & \vec{v}_2 \end{array}$$

$$(W^{\perp})^{\perp} = span\left\{ \begin{bmatrix} 2.5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -.5 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\vec{w}_2 = -2\vec{v}_2 = -2\begin{bmatrix} -.5 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{w}_1 = 1\vec{v}_1 + 3\vec{v}_2$$

wrong in class

$$\vec{z} = t_2\begin{bmatrix} 2.5 \\ 1 \\ 0 \end{bmatrix} + t_3\begin{bmatrix} -.5 \\ 0 \\ 1 \end{bmatrix}$$

$\vec{w}_1$ ✓

$$\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2.5 \\ 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} -.5 \\ 0 \\ 1 \end{bmatrix}$$

Wed Apr 4

- 6.2-6.3 very good bases revisited: orthogonal and orthonormal bases. Projection onto multi-dimensional subspaces.

Announcements: Hw: 6.2 7 9 11, 13 17
6.3 1 3, 5 7 11 13 21

- A bit in Tues. notes ~ fix a typo
- most of today's
- quiz: projection problem.

'til 12:56

Warm-up Exercise: Show that the following set of vectors is <u>orthogonal</u>: i.e. the vectors in the set are all $\perp$ to each other

$$\left\{ \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} \right\}$$

$\vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3$

$\vec{v}_1 \cdot \vec{v}_2 = 2-4+2 = 0$

$\vec{v}_1 \cdot \vec{v}_3 = -4+2+2 = 0$

$\vec{v}_2 \cdot \vec{v}_3 = -2-2+4 = 0$

<u>orthonormal</u>

orthogonal & unit length.

e.g. (standard basis vectors, e.g. in $\mathbb{R}^3$ $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$)

$$\left\{ \frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \frac{1}{3}\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, \frac{1}{3}\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} \right\}$$

$\vec{u}_1, \quad \vec{u}_2, \quad \vec{u}_3$

$\frac{\vec{v}}{\|\vec{v}\|}$ is a unit vector

Definition: The set $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_p\} \subseteq \mathbb{R}^n$ is called *orthonormal* if and only if

$$\boldsymbol{u}_i \cdot \boldsymbol{u}_i = 1, \quad i = 1, 2, \ldots p \qquad \|u_i\| = 1$$

$$\boldsymbol{u}_i \cdot \boldsymbol{u}_j = 0, \qquad i \neq j.$$

So this is a set of unit vectors that are mutually orthogonal. It turns out that they make very good bases for p-dimensional subspaces W.

Bases which are only orthogonal are also good.

Examples you know already:

1) The standard basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n\} \subseteq \mathbb{R}^n$, or any subset of the standard basis vectors.

$$\vec{e}_i \cdot \vec{e}_j = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$$

2) Rotated bases in $\mathbb{R}^2$. $\{\boldsymbol{u}_1, \boldsymbol{u}_2\} = \left\{ \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}, \begin{bmatrix} -\sin\alpha \\ \cos\alpha \end{bmatrix} \right\}$.

Theorem (why orthonormal sets are very good bases): Let $B = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_p\} \subseteq \mathbb{R}^n$ be orthonormal. Let $W = span\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_p\}$. Then

a) $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_p\}$ is linearly independent, so a basis for W.

If $c_1\vec{u}_1 + c_2\vec{u}_2 + \ldots + c_p\vec{u}_p = \vec{0}$

$\Rightarrow (c_1\vec{u}_1 + c_2\vec{u}_2 + \ldots + c_p\vec{u}_p)\cdot\vec{u}_1 = \vec{0}\cdot\vec{u}_1 = 0$

$= c_1\vec{u}_1\cdot\vec{u}_1 + c_2\vec{u}_2\cdot\vec{u}_1 + \ldots + c_p\vec{u}_p\cdot\vec{u}_1 = 0$

$c_1 + 0 + 0 + \ldots + 0 = 0 \Rightarrow c_1 = 0$

$(c_1\vec{u}_1 + c_2\vec{u}_2 + \ldots + c_p\vec{u}_p)\cdot\vec{u}_j = \vec{0}\cdot\vec{u}_j = 0 \qquad 1 \le j \le p.$

$0 + 0 + \ldots + c_j\vec{u}_j\cdot\vec{u}_j + 0 \ldots = 0$

$c_j = 0$

b) For $\boldsymbol{w} \in W$, the coordinate vector $[\boldsymbol{w}]_B = \begin{bmatrix} \boldsymbol{u}_1 \cdot \boldsymbol{w} \\ \boldsymbol{u}_2 \cdot \boldsymbol{w} \\ \vdots \\ \boldsymbol{u}_p \cdot \boldsymbol{w} \end{bmatrix}$ is directly computable. In other words,

$W = span\{\vec{u}_1, \vec{u}_2, \ldots \vec{u}_p\}$

$$\boldsymbol{w} = (\boldsymbol{u}_1 \cdot \boldsymbol{w})\boldsymbol{u}_1 + (\boldsymbol{u}_2 \cdot \boldsymbol{w})\boldsymbol{u}_2 + \ldots + (\boldsymbol{u}_p \cdot \boldsymbol{w})\boldsymbol{u}_p$$

why?

$\vec{w}\cdot\vec{u}_j = (c_1\vec{u}_1 + c_2\vec{u}_2 + \ldots + c_p\vec{u}_p)\cdot\vec{u}_j$

$= c_1\vec{u}_1\cdot\vec{u}_j + c_2\vec{u}_2\cdot\vec{u}_j + \ldots$

$= c_j$

$$\vec{u}_i \cdot \vec{u}_j = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$$

$\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ in $\mathbb{R}^3$

$\begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} = 1\vec{e}_1 + 4\vec{e}_2 + 7\vec{e}_3$

$\begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}\cdot\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}\cdot\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}\cdot\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

c) Let $\mathbf{x} \in \mathbb{R}^n$. Then there is a unique nearest point to $\mathbf{x}$ in W, which we call $proj_W\mathbf{x}$, ("the projection of $\mathbf{x}$ onto W.") The formula for this projection is given by

$W = \text{span}\{\vec{u}_1, \vec{u}_2, \ldots \vec{u}_p\}$

$$proj_W\mathbf{x} = (\boldsymbol{u}_1 \bullet \mathbf{x})\boldsymbol{u}_1 + (\boldsymbol{u}_2 \bullet \mathbf{x})\boldsymbol{u}_2 + \ldots + (\boldsymbol{u}_p \bullet \mathbf{x})\boldsymbol{u}_p .$$

(As should be the case, projection onto W leaves elements of W fixed.)

<u>Proof</u>: We will use the Pythagorean Theorem to show that the formula above for $proj_W\mathbf{x}$ yields the nearest point in W to $\mathbf{x}$:

Define

$$\mathbf{z} = \mathbf{x} - proj_W\mathbf{x}$$

$$\left[\ \mathbf{z} = \mathbf{x} - (\boldsymbol{u}_1 \bullet \mathbf{x})\boldsymbol{u}_1 - (\boldsymbol{u}_2 \bullet \mathbf{x})\boldsymbol{u}_2 - \ldots - (\boldsymbol{u}_p \bullet \mathbf{x})\boldsymbol{u}_p .\ \right] \cdot \vec{u}_j$$

Then for $j = 1, 2, \ldots p$,

$$\mathbf{z} \bullet \boldsymbol{u}_j = \mathbf{x} \bullet \boldsymbol{u}_j - \mathbf{x} \bullet \boldsymbol{u}_j = 0 .$$

$$= \vec{x}\cdot\vec{u}_j - \cancel{(\vec{u}_1\cdot\vec{x})\vec{u}_1\cdot\vec{u}_j}^{\,0} - \cancel{(\vec{u}_2\cdot\vec{x})\vec{u}_2\cdot\vec{u}_j}^{\,0} - (\vec{u}_j\cdot\vec{x})\underbrace{\vec{u}_j\cdot\vec{u}_j}_{=1} - \ldots$$

$\mathbf{z} \in W^{\perp}$

So $\mathbf{z} \perp W$, i.e.

$$\mathbf{z} \bullet (t_1\boldsymbol{u}_1 + t_2\boldsymbol{u}_2 + \ldots t_p\boldsymbol{u}_p) = 0$$

for all choices of the weight vector $\boldsymbol{t}$.

Let $w \in W$. Then

$$\|x - \mathbf{w}\|^2 = \|\underbrace{(\mathbf{x} - proj_W\mathbf{x})}_{\vec{z}} + \underbrace{(proj_W\mathbf{x} - \mathbf{w})}_{\in W}\|^2 .$$

$\|\vec{a} + \vec{b}\|^2 = \|\vec{a}\|^2 + \|\vec{b}\|^2 + 2\vec{a}\cdot\vec{b}$

Since $(\mathbf{x} - proj_W\mathbf{x}) = \mathbf{z}$ and since $(proj_W\mathbf{x} - \mathbf{w}) \in W$, we have the Pythagorean Theorem

$$\|\mathbf{x} - \mathbf{w}\|^2 = \|x - proj_W\mathbf{x}\|^2 + \|proj_W\mathbf{x} - \mathbf{w}\|^2$$

$$\|\mathbf{x} - \mathbf{w}\|^2 = \|\mathbf{z}\|^2 + \|proj_W\mathbf{x} - \mathbf{w}\|^2 .$$

So $\|\mathbf{x} - \mathbf{w}\|^2$ is always at least $\|\mathbf{z}\|^2$, with equality if and only if $\mathbf{w} = proj_W\mathbf{x}$.

QED

Exercise 1

1a) Check that the set

$$B = \left\{ \frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \frac{1}{3}\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, \frac{1}{3}\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} \right\}$$

is an orthonormal basis for $\mathbb{R}^3$.

1b) For $\underline{\boldsymbol{x}} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ find the coordinate vector $[\underline{\boldsymbol{x}}]_B$ and check your answer.

solution $[\underline{\boldsymbol{x}}]_B = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$

<u>Exercise 2</u> Consider the plane from Tuesday

$$W = span\left\{ \overset{\vec{v}_1}{\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}}, \overset{\vec{v}_2}{\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}} \right\}$$

$$W^{\perp} = span\left\{ \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix} \right\}$$

which is also given implicitly as a nullspace,

$$W = Nul\begin{bmatrix} 2 & -5 & 1 \end{bmatrix}.$$

$$\begin{bmatrix} 2 & -5 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 2 & -5 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} = 0$$

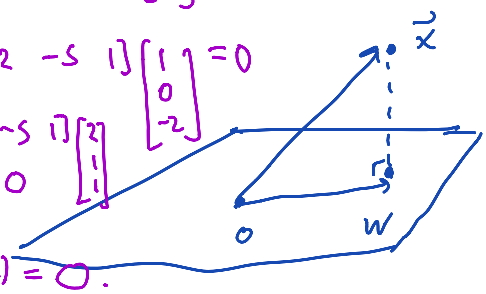
<u>2a)</u> Verify that

$$B = \left\{ \underset{\vec{u}_1}{\frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}}, \underset{\vec{u}_2}{\frac{1}{\sqrt{6}}\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}} \right\}$$

is an ortho-normal basis for W.

$$\begin{bmatrix} 2 & -5 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} = 0$$

$$\begin{bmatrix} 2 & -5 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = 0$$



$$\vec{u}_1 \cdot \vec{u}_2 = \frac{1}{\sqrt{30}}(2+0-2) = 0.$$

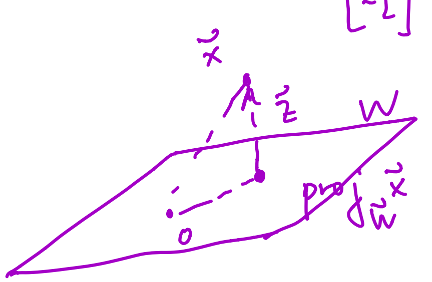
<u>2b)</u> Find $proj_W \underline{\boldsymbol{x}}$ for $\underline{\boldsymbol{x}} = \begin{bmatrix} 7 \\ -3 \\ 1 \end{bmatrix}$. Then verify that $\underline{\boldsymbol{z}} = \underline{\boldsymbol{x}} - proj_W \underline{\boldsymbol{x}}$ is perpendicular to W.

solution $proj_W \underline{\boldsymbol{x}} = \begin{bmatrix} 5 \\ 2 \\ 0 \end{bmatrix}$

$$proj_W \vec{x} = (\vec{x}\cdot\vec{u}_1)\vec{u}_1 + (\vec{x}\cdot\vec{u}_2)\vec{u}_2$$

$$= \left(\begin{bmatrix} 7 \\ -3 \\ 1 \end{bmatrix} \cdot \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}\right)\frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + \left(\begin{bmatrix} 7 \\ -3 \\ 1 \end{bmatrix} \cdot \frac{1}{\sqrt{6}}\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}\right)\frac{1}{\sqrt{6}}\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{5}\cdot 5\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + \frac{1}{6}\,12\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 0 \end{bmatrix}$$



$$\begin{bmatrix} 7 \\ -3 \\ 1 \end{bmatrix} - \begin{bmatrix} 5 \\ 2 \\ 0 \end{bmatrix}$$

$$\vec{x} = proj_W \vec{x} + \underbrace{(\vec{x} - proj_W \vec{x})}_{\text{"}\vec{z}\text{"}}$$

$$\vec{x} = \begin{bmatrix} 5 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}$$

$$\vec{x} = \underset{\in W}{\tilde{w}} + \underset{\in W^{\perp}}{\vec{z}}$$

Remark: A basis $\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_p\} \subseteq \mathbb{R}^n$ is called *orthogonal* if the the vectors in the set are mutually perpendicular, but not necessarily normalized to unit length. One can construct an orthonormal basis from that set by normalizing, namely

$$\{\underline{u}_1, \underline{u}_2, \dots, \underline{u}_p\} = \left\{ \frac{\underline{v}_1}{\|\underline{v}_1\|}, \frac{\underline{v}_2}{\|\underline{v}_2\|}, \dots, \frac{\underline{v}_p}{\|\underline{v}_p\|} \right\} \subseteq \mathbb{R}^n .$$

One can avoid square roots if one uses the original orthogonal matrix rather than the ortho-normal one. This is the approach the text prefers. For example, for orthogonal bases, the very good basis theorem reads

Theorem (why orthogonal bases are very good bases): Let $B = \{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_p\} \subseteq \mathbb{R}^n$ be orthogonal. Let $W = span\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_p\}$. Then

a) $\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_p\}$ is linearly independent, so a basis for W.

b) For $\underline{w} \in W$,

$$\underline{w} = (\underline{u}_1 \bullet \underline{w})\underline{u}_1 + (\underline{u}_2 \bullet \underline{w})\underline{u}_2 + \dots + (\underline{u}_p \bullet \underline{w})\underline{u}_p$$

$$\underline{w} = \frac{(\underline{v}_1 \bullet \underline{w})}{\|\underline{v}_1\|^2}\underline{v}_1 + \frac{(\underline{v}_2 \bullet \underline{w})}{\|\underline{v}_2\|^2}\underline{v}_2 + \dots + \frac{(\underline{v}_p \bullet \underline{w})}{\|\underline{v}_p\|^2}\underline{v}_p$$

c) Let $\underline{x} \in \mathbb{R}^n$. Then there is a unique nearest point to $\underline{x}$ in W, which we call $proj_W\underline{x}$, ("the projection of $\underline{x}$ onto W.") The formula for this projection is given by

$$proj_W\underline{x} = (\underline{u}_1 \bullet \underline{x})\underline{u}_1 + (\underline{u}_2 \bullet \underline{x})\underline{u}_2 + \dots + (\underline{u}_p \bullet \underline{x})\underline{u}_p .$$

$$proj_W\underline{x} = \frac{(\underline{v}_1 \bullet \underline{x})}{\|\underline{v}_1\|^2}\underline{v}_1 + \frac{(\underline{v}_2 \bullet \underline{x})}{\|\underline{v}_2\|^2}\underline{v}_2 + \dots + \frac{(\underline{v}_p \bullet \underline{x})}{\|\underline{v}_p\|^2}\underline{v}_p.$$

You can see how that would have played out in the previous exercise.