2h) Refer to the same diagram as in 2g, which is an $\mathbb{R}^n$ picture. Using the Pythagorean triangle with edges $(\underline{\boldsymbol{x}} \cdot \underline{\boldsymbol{u}})\underline{\boldsymbol{u}}$, $\underline{z}$, $\underline{\boldsymbol{x}}$ we have

$$\| \underbrace{(\underline{\boldsymbol{x}} \cdot \underline{\boldsymbol{u}})\underline{\boldsymbol{u}}}_{\text{proj}_L \vec{x}} \|^2 + \|\underline{z}\|^2 = \|\underline{\boldsymbol{x}}\|^2.$$

Define the angle θ between $\underline{v}$ and $\underline{w}$ the same way we would in $\mathbb{R}^2$, namely

$$\cos(\theta) = \frac{(\underline{\boldsymbol{x}} \cdot \underline{\boldsymbol{u}})}{\|\underline{\boldsymbol{x}}\|}.$$

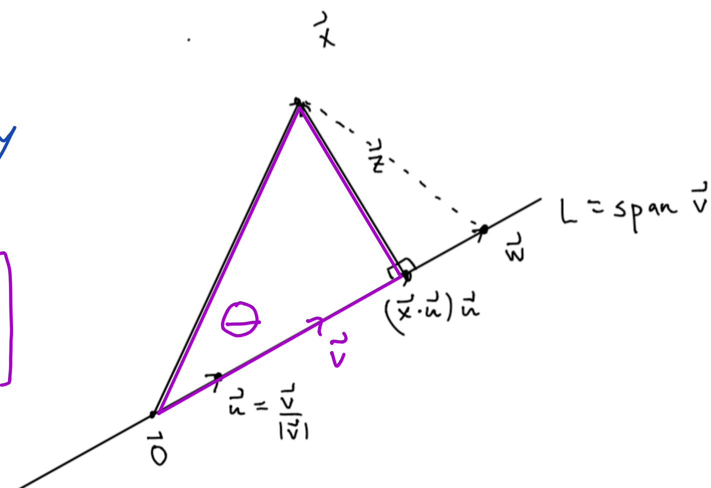
Notice that because of the Pythagorean identity above, $-1 \le \cos(\theta) \le 1$, with $\cos(\theta) = 1$ if and only if $(\underline{\boldsymbol{x}} \cdot \underline{\boldsymbol{u}})\underline{\boldsymbol{u}} = \underline{\boldsymbol{x}}$ and $\cos(\theta) = -1$ if and only if $(\underline{\boldsymbol{x}} \cdot \underline{\boldsymbol{u}})\underline{\boldsymbol{u}} = -\underline{\boldsymbol{x}}$. So there is a unique θ with $0 \le \theta \le \pi$ for whic the cos θ equation can hold. Substituting $u = \dfrac{v}{\|v\|}$ gives the familiar formulas that you learned in multivariable Calculus for $\mathbb{R}^2$, $\mathbb{R}^3$, which now holds in $\mathbb{R}^n$.

$$\cos(\theta) = \frac{\left(\underline{\boldsymbol{x}} \cdot \dfrac{\underline{\boldsymbol{v}}}{\|\underline{\boldsymbol{v}}\|}\right)}{\|\underline{\boldsymbol{x}}\|} = \frac{(\underline{\boldsymbol{x}} \cdot \underline{\boldsymbol{v}})}{\|\underline{\boldsymbol{x}}\| \, \|\underline{\boldsymbol{v}}\|}, \text{ i.e.}$$

$$(\underline{\boldsymbol{x}} \cdot \underline{\boldsymbol{v}}) = \|\underline{\boldsymbol{x}}\| \, \|\underline{\boldsymbol{v}}\| \cos(\theta)$$

we'll discuss this on Tuesday

define "θ" in $\mathbb{R}^n$.



Ex $\vec{x} = \begin{bmatrix}1\\2\\3\\4\end{bmatrix}$, $\vec{v} = \begin{bmatrix}1\\0\\3\\0\end{bmatrix}$

$$\cos\theta = \frac{\vec{x}\cdot\vec{v}}{\|\vec{x}\|\,\|\vec{v}\|} = \frac{10}{\sqrt{30}\,\sqrt{10}} = \frac{10}{10\sqrt{3}} = \frac{1}{\sqrt{3}}$$

$0 < \theta < \pi/2$.

$\vec{z}: \vec{x} - (\vec{x}\cdot\vec{u})\vec{u}$

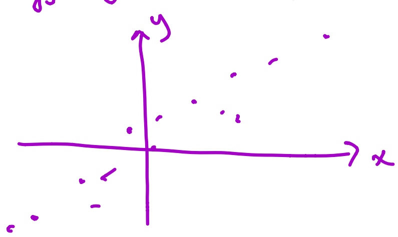
$\vec{z}\cdot\vec{u} = (\vec{x} - (\vec{x}\cdot\vec{u})\vec{u})\cdot\vec{u} = (\vec{x}\cdot\vec{u}) - (\vec{x}\cdot\vec{u}) = 0$

so $\vec{z}\cdot t\vec{u} = 0 \ \forall t$

for $\vec{w} \in L$, $\|\vec{x}-\vec{w}\|^2 = \|\vec{z}\|^2 + \|\vec{w} - (\vec{x}\cdot\vec{u})\vec{u}\|^2$
$\ge \|\vec{z}\|^2$
$= \|\vec{z}\|^2$ iff $\vec{w} = (\vec{x}\cdot\vec{u})\vec{u}$

Stats!

$\{(x_j, y_j),\ j = 1, 2, \dots N\}$



"correlation coeff" r, $-1 \le r \le 1$

neg. cor. highly positively correlated

$$r = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{n}(x_i-\bar{x})^2}\sqrt{\sum_{i=1}^{n}(y_i-\bar{y})^2}}$$

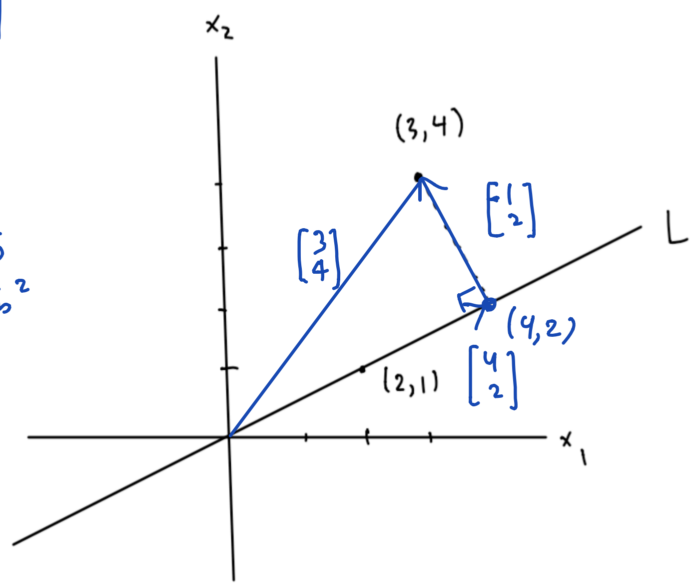
$\bar{x}, \bar{y}$ averages!

$r = \cos\theta$ for $\begin{bmatrix} x_1-\bar{x} \\ x_2-\bar{x} \\ \vdots \\ x_n-\bar{x} \end{bmatrix}, \begin{bmatrix} y_1-\bar{y} \\ y_2-\bar{y} \\ \vdots \\ y_n-\bar{y} \end{bmatrix}$

<u>3) Summary exercise</u> In $\mathbb{R}^2$, let $L = span\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$. Find $proj_L\begin{bmatrix} 3 \\ 4 \end{bmatrix}$. Illustrate. Verify the Pythagorean Theorem for $proj_L\begin{bmatrix} 3 \\ 4 \end{bmatrix}$, "$\underline{z}$" and hypotenuse $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

$$\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \vec{u} = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\begin{aligned} proj_L \vec{x} &= (\vec{x}\cdot\vec{u})\,\vec{u} \\ &= \left(\begin{bmatrix} 3 \\ 4 \end{bmatrix} \cdot \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right)\frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix} \\ &= \frac{1}{5}\left(\begin{bmatrix} 3 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right)\begin{bmatrix} 2 \\ 1 \end{bmatrix} \\ &= \frac{10}{5}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \end{aligned}$$



Pythag: $25 = 20 + 5$
$c^2 = a^2 + b^2$

notice also
$$\vec{z} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \perp \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 0$$

Tues Apr 3

- 6.1-6.2 Orthogonal complements to subspaces, and the four fundamental subspace theorem revisited.

Announcements:
- finish M. notes
- today, revisit 4 subspaces associated to a matrix
  do the geometric part right.
- quiz tomorrow will be projection problem.

'til 12:56

Warm-up Exercise: Let $L = \text{span}\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\} = \text{span}\{\vec{v}\}$

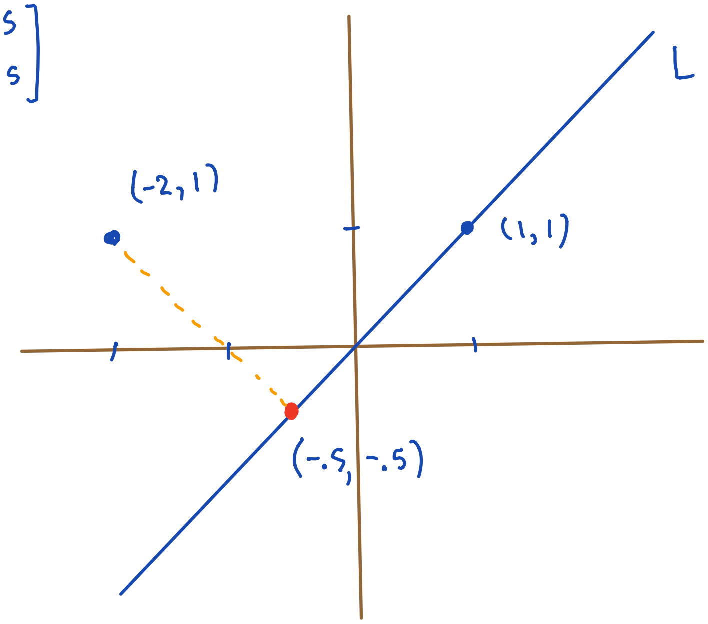
Let $\vec{x} = \begin{bmatrix}-2\\1\end{bmatrix}$

Find $\text{proj}_L \vec{x} = (\vec{x}\cdot\vec{u})\vec{u} = \begin{bmatrix}-.5\\-.5\end{bmatrix}$

where $\vec{u}$ is the unit vector in the direction of $\vec{v}$.

$\vec{v} = \begin{bmatrix}1\\1\end{bmatrix}$, $\vec{u} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$

$$\text{proj}_L \vec{x} = \left(\begin{bmatrix}-2\\1\end{bmatrix}\cdot\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}\right)\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$$
$$= \frac{1}{2}(-2+1)\begin{bmatrix}1\\1\end{bmatrix}$$
$$= -\frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix}$$



Note $\vec{u} = \frac{\vec{v}}{\|\vec{v}\|}$

Prof. K. likes

$$\text{proj}_L \vec{x}: \quad (\vec{x}\cdot\vec{u})\vec{u} = \left(\vec{x}\cdot\frac{\vec{v}}{\|\vec{v}\|}\right)\frac{\vec{v}}{\|\vec{v}\|}$$
$$= \frac{\vec{x}\cdot\vec{v}}{\|\vec{v}\|^2}\vec{v}$$

text likes this one.
no square roots.

Orthogonal complements, and the four subspaces associated with a matrix transformation, revisited more carefully than our first time through.

Let $W \subseteq \mathbb{R}^n$ be a subspace of dimension $1 \le p \le n$. The *orthogonal complement to* W is the collection of all vectors perpendicular to every vector in W. We write the orthogonal complement to W as $W^{\perp}$, and say "W perp". Let $B = \{\underline{\boldsymbol{w}}_1, \underline{\boldsymbol{w}}_2, \ldots \underline{\boldsymbol{w}}_p\}$ be a basis for W. Let $\underline{\boldsymbol{v}} \in W^{\perp}$. This means

$$\left(c_1 \boldsymbol{w}_1 + c_2 \underline{\boldsymbol{w}}_2 + \ldots + c_p \underline{\boldsymbol{w}}_p\right) \cdot \underline{\boldsymbol{v}} = 0$$

for all linear combinations of the spanning vectors. Since the dot product distributes over linear combinations, the identity above expands as

$$c_1\left(\underline{\boldsymbol{w}}_1 \cdot \underline{\boldsymbol{v}}\right) + c_2\left(\underline{\boldsymbol{w}}_2 \cdot \underline{\boldsymbol{v}}\right) + \ldots + c_p\left(\underline{\boldsymbol{w}}_p \cdot \underline{\boldsymbol{v}}\right) = 0$$

for all possible weights. This is true if and only if

$$\underline{\boldsymbol{w}}_1 \cdot \underline{\boldsymbol{v}} = \underline{\boldsymbol{w}}_2 \cdot \underline{\boldsymbol{v}} = \ldots = \underline{\boldsymbol{w}}_p \cdot \underline{\boldsymbol{v}} = 0.$$

In other words, $\underline{\boldsymbol{v}} \in Nul\, A$ where A is the $m \times n$ matrix having the spanning vectors as rows:

$$A\,\underline{\boldsymbol{v}} = \begin{bmatrix} \underline{\boldsymbol{w}}_1^T \\ \underline{\boldsymbol{w}}_2^T \\ \vdots \\ \underline{\boldsymbol{w}}_p^T \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \underline{\boldsymbol{0}}.$$

So

$$W^{\perp} = Nul\, A.$$

<u>Exercise 1</u> Find $W^{\perp}$ for $W = span\left\{\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}\right\}$.

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} \vec{v} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 1 & 0 & -2 & 0 \end{array}\right]$$

$$\begin{array}{r} \\ -R_1 + R_2 \to R_2 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -1 & -5 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & 1 & 5 & 0 \end{array}\right]$$

$$\begin{array}{r} -R_2 + R_1 \\ \end{array} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 5 & 0 \end{array}\right]$$

$$W^{\perp} = span\left\{\begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}\right\}$$

check: $\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix} = 2 - 5 + 3 = 0$!

$\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix} = 2 + 0 - 2 = 0$!

Theorem (fill in details).

$1 \le p \le n-1$, $0 \le p \le n$

1a) Let $W \subseteq \mathbb{R}^n$ be a subspace with $dim\ W = p$, $1 \le p \le n$. Then $dim(W^{\perp}) = n - p$, so

$$dim(W) + dim(W^{\perp}) = n$$

Hint: Use reduced row echelon form ideas.

Let $A = \underbrace{\begin{bmatrix} \vec{w}_1^T \\ \vec{w}_2^T \\ \vdots \\ \vec{w}_p^T \end{bmatrix}}_{n} \Big\} p$ where $\{\vec{w}_1, \vec{w}_2, \ldots \vec{w}_p\}$ basis for W.

rref(A) has p pivots
otherwise elementary row ops would've created a basis for W with $< p$ elements, but $\dim W = p$

$W^{\perp} = \mathrm{Nul}\,A$, $\dim(\mathrm{Nul}\,A) = n-p$
$=$ # of non-pivot columns.

1b) $W \cap W^{\perp} = \{\underline{\mathbf{0}}\}$

Hint: Let $\underline{\boldsymbol{x}} \in W \cap W^{\perp}$. Compute $\underline{\boldsymbol{x}} \cdot \underline{\boldsymbol{x}}$.

intersection

$\underset{W}{\underset{\cap}{\vec{x}}} \cdot \underset{W^{\perp}}{\underset{\cap}{\vec{x}}} = 0$ because $\vec{x} \in W^{\perp}$, $\vec{x} \cdot \vec{w} = \vec{0}$ $\forall \vec{w} \in W$

1c) $(W^{\perp})^{\perp} = W$.

Hint: Show $W \subseteq (W^{\perp})^{\perp}$. Then count dimensions.

$\vec{u} \in (W^{\perp})^{\perp}$ means $\vec{u} \cdot \vec{z} = 0$ $\forall \vec{z} \in W^{\perp}$

if $\vec{w} \in W$, and $\vec{z} \in W^{\perp}$ then $\vec{w} \cdot \vec{z} = 0$

So $\vec{w} \in (W^{\perp})^{\perp}$

$\dim W + \dim W^{\perp} = n$

$\Rightarrow \dim W^{\perp} + (\dim W^{\perp})^{\perp} = n$

$\Rightarrow \dim W = \dim (W^{\perp})^{\perp}$

but $W \subset (W^{\perp})^{\perp}$

Since dim's are $=$

$W = (W^{\perp})^{\perp}$

(the q-dim'l subspace of q-dim'l vectorspace is entire vector space)

1d) Let $B = \{\underline{\boldsymbol{w}}_1, \underline{\boldsymbol{w}}_2, \ldots \underline{\boldsymbol{w}}_p\}$ be a basis for W and $C = \{\underline{\boldsymbol{z}}_1, \underline{\boldsymbol{z}}_2, \ldots \underline{\boldsymbol{z}}_{n-p}\}$ be a basis for $W^{\perp}$. Then their union, $B \cup C$, is a basis for $\mathbb{R}^n$.

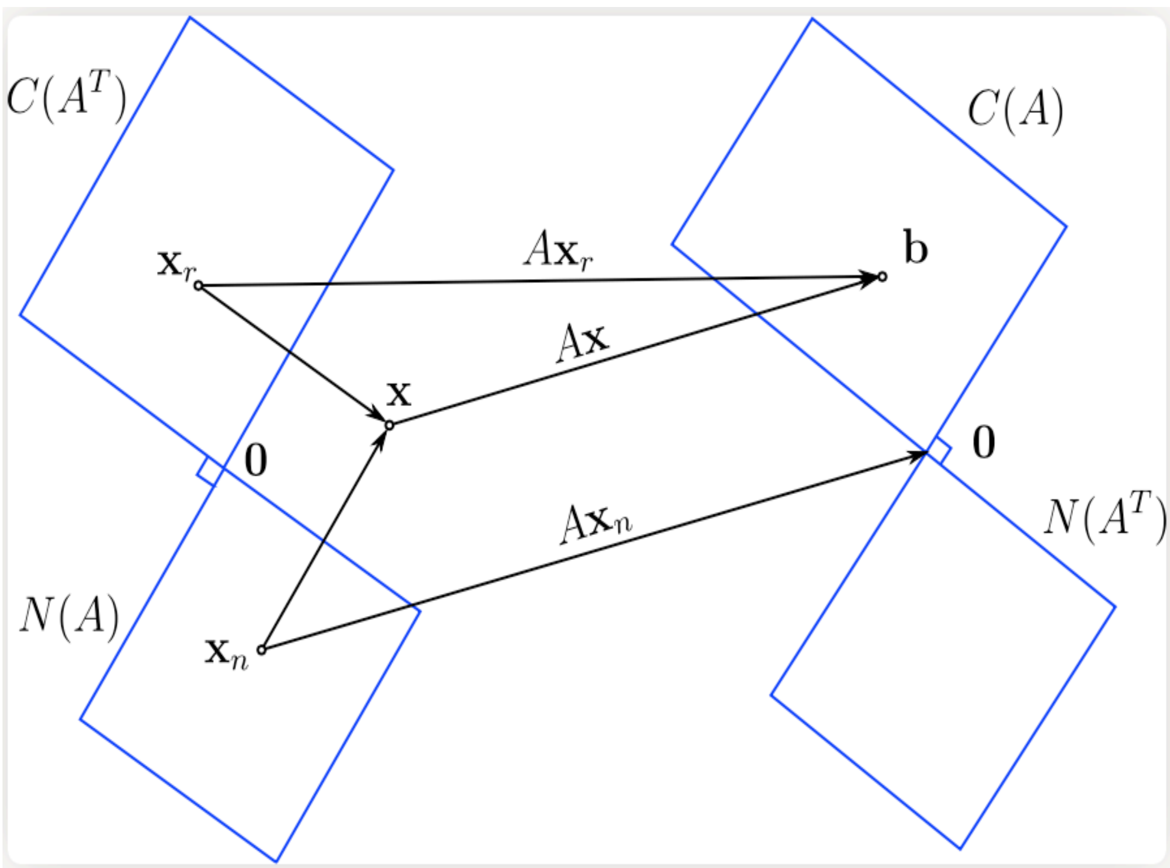
Hint: Show $B \cup C$ is linearly independent.

next time!

Remark: From the discussion above, and for any $m \times n$ matrix A of arbitrary rank p, we can deduce from the discussion above that $(Row\ A)^{\perp} = Nul\ A$; so $(Nul\ A)^{\perp} = Row\ A$; from our previous work we know that $dim\ (Row\ A) = p$, $dim\,(Nul\ A) = n - p$. This decomposes the domain of the linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$,

$$T(\underline{\boldsymbol{x}}) := A\,\underline{\boldsymbol{x}}\,.$$

By the same reasoning applied to the transpose transformation from $\mathbb{R}^m \to \mathbb{R}^n$, the codomain of T decomposes into $Col\ A = Row\ A^T$ and $(Col\ A)^{\perp} = Nul\ A^T$, with $dim\,(Col\ A) = p$ and $dim\ \left(Nul\ A^T\right) = m - p$. In other words, we have justified the diagram we really only waved our hands at back in Chapter 4, except for transformations from $\mathbb{R}^2 \to \mathbb{R}^2$.



$$\begin{array}{cc} \vec{w}_1 & \vec{w}_2 \end{array}$$

Exercise 2) In Exercise 1 with $W = span\left\{\begin{bmatrix}1\\1\\3\end{bmatrix}, \begin{bmatrix}1\\0\\-2\end{bmatrix}\right\}$, we showed $W^{\perp} = span\left\{\begin{bmatrix}2\\-5\\1\end{bmatrix}\right\}$. Compute $(W^{\perp})^{\perp}$ as $Nul\begin{bmatrix}2 & -5 & 1\end{bmatrix}$ and verify that it recovers W (but with a different basis).

$$\begin{bmatrix}2 & -5 & 1\end{bmatrix}\begin{bmatrix}z_1\\z_2\\z_3\end{bmatrix} = 0$$

$$\left[\begin{array}{ccc|c} 2 & -5 & 1 & 0 \\ \hline 1 & -2.5 & .5 & 0 \end{array}\right]$$

$z_1 = 2.5t_2 - .5t_3$
$z_2 = t_2$
$z_3 = t_3$

$$\vec{z} = t_2\begin{bmatrix}2.5\\1\\0\end{bmatrix} + t_3\begin{bmatrix}-.5\\0\\1\end{bmatrix}$$

$$\begin{array}{cc} \vec{v}_1 & \vec{v}_2 \end{array}$$

$$(W^{\perp})^{\perp} = span\left\{\begin{bmatrix}2.5\\1\\0\end{bmatrix}, \begin{bmatrix}-.5\\0\\1\end{bmatrix}\right\}$$

$\vec{w}_2 = -2\vec{v}_2$

$\vec{w}_1 = 1\vec{v}_1 + 3\vec{v}_2$

wrong in class