Free index cards, use 1 on exam.

regular room WEB L110

Math 2270-004
Final Exam Review Information
April 24, 2018

also, by having you explain reasoning or concepts

Our final exam is next Monday afternoon, April 30, 1:00-3:00 p.m., in our usual classroom WEB L110. I will let you work until 3:30 p.m. if you wish. There will be a review session this Friday April 27, from 2:00-4:00 p.m., with room TBA. Most of that session will be devoted to going over a practice exam (which I will try to post by Wednesday evening), but please bring any other questions you may have.

The exam will be comprehensive. Precisely, you can expect anything we covered from sections 1.1-1.9, 2.1-2.3, 3.1-3.3, 4.1-4.7, 5.1-5.6, 6.1-6.8, 7.1-7.2 - we'll discuss in more detail below. In addition to being able to do computations, you should know key definitions, the statements of the main theorems, and why they are true. The exam will be a mixture of computational and conceptual questions. As on the midterms, I will ~~primarily~~ test conceptual understanding ~~with true-false and "example" questions~~. The exam is closed book and closed note, except for a single index card of size up to ~~5 by 7~~ 5×8 inches. You may use a scientific calculator to check arithmetic, if you wish.

Exam material will be weighted towards topics which have not yet been tested, i.e. 5.4-5.5 and chapter 6-7 material.

Copies of my final exams from previous years can be found on my "old classes" web page, although we used a different textbook and had somewhat different emphases back then. This semester's midterms, quizzes, homework, class notes, and the text are good references. It always worked well for me as a student to make my own course outline with the key ideas (which I would then make sure I could explain and work with).

Learning Objectives for 2270

Computation vs. Theory: This course is a combination of computational mathematics and theoretical mathematics. By theoretical mathematics, I mean abstract definitions and theorems, instead of calculations. The computational aspects of the course may feel more familiar and easier to grasp, but I urge you to focus on the theoretical aspects of the subject. Linear algebra is a tool that is heavily used in mathematics, engineering, and science, so it will likely be relevant to you later in your career. When this time comes, you will find that the computations of linear algebra can easily be done by computing systems such as Matlab, Maple, Mathematica or Wolfram alpha. But to understand the significance of these computations, a person must understand the theory of linear algebra. Understanding abstract mathematics is something that comes with practice, and takes more time than repeating a calculation. When you encounter an abstract concept in lecture, I encourage you to pause and give yourself some time to think about it. Try to give examples of the concept, and think about what the concept is good for.

The essential topics

Be able to find the solution set to linear systems of equations systematically, using row reduction techniques and reduced row echelon form - by hand for smaller systems and using technology for larger ones. Be able to solve (linear combination) vector equations using the same methods, as both concepts are united by the common matrix equation $A\mathbf{x} = \mathbf{b}$. rref problem.

Be able to use the correspondence between matrices and linear transformations - first for transformations between $\mathbb{R}^n$ and $\mathbb{R}^m$, and later for transformations between arbitrary vector spaces.

Become fluent in matrix algebra techniques built out of matrix addition and multiplication, in order to solve matrix equations.

Understand the algebra and geometry of determinants so that you can compute determinants, with applications to matrix inverses and to oriented volume expansion factors for linear transformations.

Become fluent in the language and concepts related to general vector spaces: linear independence, span, basis, dimension, and rank for linear transformations. Understand how change of basis in the domain and range effect the matrix of a linear transformation.

Be able to find eigenvalues and eigenvectors for square matrices. Apply these matrix algebra concepts and matrix diagonalization to understand the geometry of linear transformations and certain discrete dynamical systems.

Understand how orthogonality and angles in $\mathbf{R}^2, \mathbf{R}^3$ generalize via the dot product to $\mathbf{R}^n$, and via general inner products to other vector spaces. Be able to use orthogonal projections and the Gram-Schmidt process, with applications to least squares problems and to function vector spaces.

Know the spectral theorem for symmetric matrices and be able to find their diagonalizations. Relate this to quadratic forms, constrained optimization problems, and to the singular value decomposition for matrices. Learn some applications to image processing and/or statistics.

nope §7.3 PCA

SVD
CS people need this
(it uses spectral theorem).
they'll teach you
at the time

Topics/concepts list for final exam

Sections 1.1-1.9, 2.1-2.3, the material on the first midterm:

The matrix equation $A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}$ arises in a number of different contexts: it can represent a system of linear equations in the unknown $\underline{\boldsymbol{x}}$; a vector linear combination equation of the columns of A, with weights given by the entries of $\underline{\boldsymbol{x}}$; in the study of the linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ given by $T(\underline{\boldsymbol{x}}) = A\,\underline{\boldsymbol{x}}$.

$A = [T(\vec{e}_1) \mid T(\vec{e}_2) \mid \ldots]$

An essential tool is the reduced row echelon form of a matrix (augmented or unaugmented), and what it tells you about (1) solutions to matrix equations; (2) the structure of solution sets to matrix equations; (3) column dependencies of a matrix. (We turned the column dependency idea around to understand why each matrix can have only one reduced row echelon form.) We also discussed matrix transformations $T(x) = A\,x$ from $\mathbb{R}^n$ to $\mathbb{R}^m$, as a prelude to general linear transformations $T : V \to W$ that we discussed later in the course and that appear in subsequent courses. We focused on the geometry of linear transformations from $\mathbb{R}^2 \to \mathbb{R}^2$, $\mathbb{R}^2 \to \mathbb{R}^3$, $\mathbb{R}^3 \to \mathbb{R}^2$, etc., which are important in their own right, and where visualization helps develop intuition.

$T : \mathbb{R}^2 \to \mathbb{R}^2$ $\quad A = [T(\vec{e}_1)\ T(\vec{e}_2)]$ ← special case of matrix for a linear transfor.

Square matrices and linear transformations from $\mathbb{R}^n \to \mathbb{R}^n$ are an important special case. Invertible matrices are useful in matrix algebra computations. One should know how to find matrix inverses and use them.

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material after the first midterm

3.1-3.3 determinants.

how to compute via cofactor expansions or elementary row (or column) operations

$|A| \neq 0$ as a test for matrix invertibility, (or equivalently whether the columns of the matrix are a basis for $\mathbb{R}^n$, whether $rref(A) = I$, or anything else on the long list of equivalent characterizations).

$|det(A)|$ as area/volume expansion factor for $T(\boldsymbol{x}) = A\,\underline{x}$.

Adjoint formula for A^{-1}

abstract & important

4.1 vector spaces and sub vector spaces (subspaces) - abstract definitions.
- realization of subspaces as null spaces or as spans of collections of vectors
- how to check if a subset is a subspace.
- examples such as polynomial vector spaces, matrix vector spaces, $\mathbb{R}^n$, and subspaces of all of these.

4.2 *Nul A* and *Col A* for $T(\underline{\boldsymbol{x}}) = A\,\underline{\boldsymbol{x}}$; *Kernel T* and *Range T* for general linear transformations $T : V \to W$
- definition of linear transformation, examples.
- how to find *Nul A* and *Col A*, and bases for each.

4.3 linearly independent/dependent sets; bases for vector spaces (including subspaces).
- how to check whether the vectors in a set span a vector space.
- how to check whether a set of vectors is linearly independent.
- how to build up bases as growing sets of independent vectors, one vector at a time, until the set spans.
- how to cull dependent vectors from a spanning set, until it is an independent set.

4.4 every basis of n vectors for a vector space V yields a coordinate system, via the coordinate isomorphism with $\mathbb{R}^n$.

answering questions about span and linear independence for sets of vectors in V by using coordinates with respect to a basis.

favorite examples include P_n, $M_{m \times n}$, the polynomial and matrix spaces.

4.5 dimension of a vector space. basic facts about dimension, number of vectors required to span, maximum number of independent vectors, dimensions of subspaces.

4.6 rank of a matrix. rank + nullity theorem.

connection to reduced row echelon form of the matrix.

how to find $Row\ A$, $Nul\ A^T$.

what $Nul\ A$, $Row\ A$, $Col\ A$, $Nul\ A^T$ have to do with the geometry of the transformation $T(\underline{\mathbf{x}}) = A\,\underline{\mathbf{x}}$.

4.7 change of basis in $\mathbb{R}^n$ and in vector spaces V. (favorite examples besides $\mathbb{R}^n$ are P_n, $M_{m \times n}$).

how to find and use the change of coordinates matrix $P_{C \leftarrow B}$.

5.1-5.3 eigenvectors, eigenvalues, diagonalization

for sure

characteristic polynomial to find eigenvalues of a matrix

$E_{\lambda = \lambda_i} = Nul\ \left(A - \lambda_i I\right)$. Finding eigenvector as weights for column dependencies, or the "old" way via backsolving.

diagonalizable and non-diagonalizable matrices.

Using $A = PDP^{-1}$ to compute large matrix powers.

improved understanding of the transformation $T(\underline{\mathbf{x}}) = A\,\underline{\mathbf{x}}$ in terms of $\mathbb{R}^n$ basis made out of eigenvectors, as compared to the standard basis.

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material after the second midterm:

more heavily weighted

5.4 matrix of a linear transformation $T: V \to W$. , $\beta = \{\vec{v}_1, \vec{v}_2, .. \vec{v}_n\}$ for V, $C = \{\vec{w}_1, \vec{w}_2, ... \vec{w}_m\}$ for W

Finding the matrix of a linear transformation, given bases for V, W

What the columns of such a matrix must be

$A = \left[\quad \quad \right]$

special cases:

change of basis, with identity map $I: V \to V$

matrix of $T: \mathbb{R}^n \to \mathbb{R}^n$, $T(x) = A\,x$ with respect to an eigenbasis of A or a "better" basis than the standard one.

similar matrices

$$A\,[\vec{x}]_\beta = [T(\vec{x})]_C$$

$$\underline{col_1(A)} = A \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \underline{[T(\vec{v}_1)]_C}$$

$\leftarrow [\vec{v}_1]_\beta$

~~5.5 complex eigendata~~

5.6 discrete dynamical systems

how to write explicit formulas for $A^n \underline{\mathbf{x}}_0$ by expressing $\underline{\mathbf{x}}_0$ as a linear combination of eigenvectors

understanding limiting behavior of $A^n \underline{\mathbf{x}}_0$ from such expressions.

possible ...

$$A = \left[[T(\vec{v}_1)]_C \ [T(\vec{v}_2)]_C \cdots [T(\vec{v}_n)]_C \right]$$

lots of Chapter 6

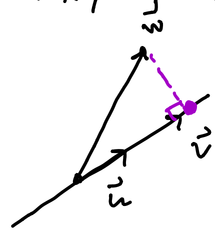
6.1 dot product (inner product), length, orthogonality.

- algebra of dot product
- Pythagorean Theorem $\|\vec{x}+\vec{y}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2$, iff $\vec{x}\cdot\vec{y} = 0$ (def $\vec{x} \perp \vec{y}$).

  $proj_L\, \underline{w}$ where $L = span\{\underline{v}\}$

  $\underline{z} = \underline{w} - proj_L\, \underline{w} \perp \underline{v}$.

  angles in $\mathbb{R}^n$



$proj_L \vec{w} = (\vec{w}\cdot\vec{u})\vec{u}$.

$\vec{z} = \vec{w} - proj_L \vec{w}$

$\vec{z}\cdot\vec{u} = 0$

orthogonal complements and how to find them
four fundamental subspaces of a matrix, revisited

6.2-6.3 orthogonal and ortho-normal sets in $\mathbb{R}^n$.

- coordinates in a subspace with respect to an orthonormal (or orthogonal basis).
- projection onto a subspace having an orthonormal (or orthogonal) basis.

$P \to Q$, $\vec{Q}-\vec{P}$

$\{\vec{u}_1, \vec{u}_2, \ldots \vec{u}_k\}$ o.n. basis for W

$\vec{w} \in W$, $\vec{w} = (\vec{w}\cdot\vec{u}_1)\vec{u}_1 + (\vec{w}\cdot\vec{u}_2)u_2 + \cdots (\vec{w}\cdot\vec{u}_k)\vec{u}_k$

$\vec{v} \in \mathbb{R}^n$ $proj_W \vec{v} = (\vec{v}\cdot\vec{u}_1)\vec{u}_1 + \cdots + (\vec{v}\cdot\vec{u}_k)\vec{u}_k$

6.4 Gram-Schmidt orthogonalization • for sure
the algorithm, algebraically and geometrically
A= QR factorization → won't ask.

6.5 Least-squares solutions to inconsistent systems • for sure.
geometric meaning, and computation via orthonormal basis for *Col A*
alternate solution using normal equations, and why this method works
computing projections without having an orthonormal basis, but just a basis.

$A\vec{x} = \vec{b}$ inconsistent

$A\hat{x} = proj_{Col A} \vec{b}$

$A^T A \vec{x} = A^T \vec{b}$ slick way.

6.6 least squares solutions for "linear models" via the normal equations
esp. best-line fit to a collection of data points

$y_i = mx_i + b$.

$$\begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \\ x_n & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

6.7-6.8 inner product spaces
Fourier series (I would provide formulas for Fourier coefficients if necessary)
probably not.

7.1 Spectral theorem for symmetric matrices
constructing orthonormal eigenbases for symmetric matrices
outer product method of multiplying matrices
spectral decomposition theorem

for sure

good problems & diagonalization of quadratic forms

7.2 quadratic forms
expressed via a symmetric matrix
diagonalizing symmetric forms with orthogonal change of variables
applications to conics and quartic surfaces ← probably not make you graph (but maybe identify)
positive definite and negative definite quadratic forms.

fluency in the defintions and concepts
ability to create examples illustrating definitions and concepts
ability to discern whether statements are true or false, based on the material we've covered.