Do algebra!

$$2x^2 + 2y^2 - 5xy = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 2 & \frac{5}{2} \\ \frac{5}{2} & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} x' & y' \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & +\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 2 & \frac{5}{2} \\ \frac{5}{2} & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$= \begin{bmatrix} x' & y' \end{bmatrix} \begin{bmatrix} \frac{9}{2} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} \quad \text{(because } P^T AP = \text{D)}$$

$$= \frac{9}{2}(x')^2 - \frac{1}{2}(y')^2.$$

$\vec{x}^T A \vec{x}$

$\vec{x} = P\vec{y}$

$\vec{y}^T \underbrace{P^T A P}_{= D} \vec{y}$

So the original curve with equation

$$2x^2 + 2y^2 + 5xy = 1 \qquad \text{standard coords}$$

in the standard coordinate system has equation

$$\frac{9}{2}(x')^2 - \frac{1}{2}(y')^2 = 1 \qquad \text{B-coords}$$

with respect to the rotated coordinate system!

Answer to 1a) This curve is a hyperbola! In the rotated coordinate system its equation is

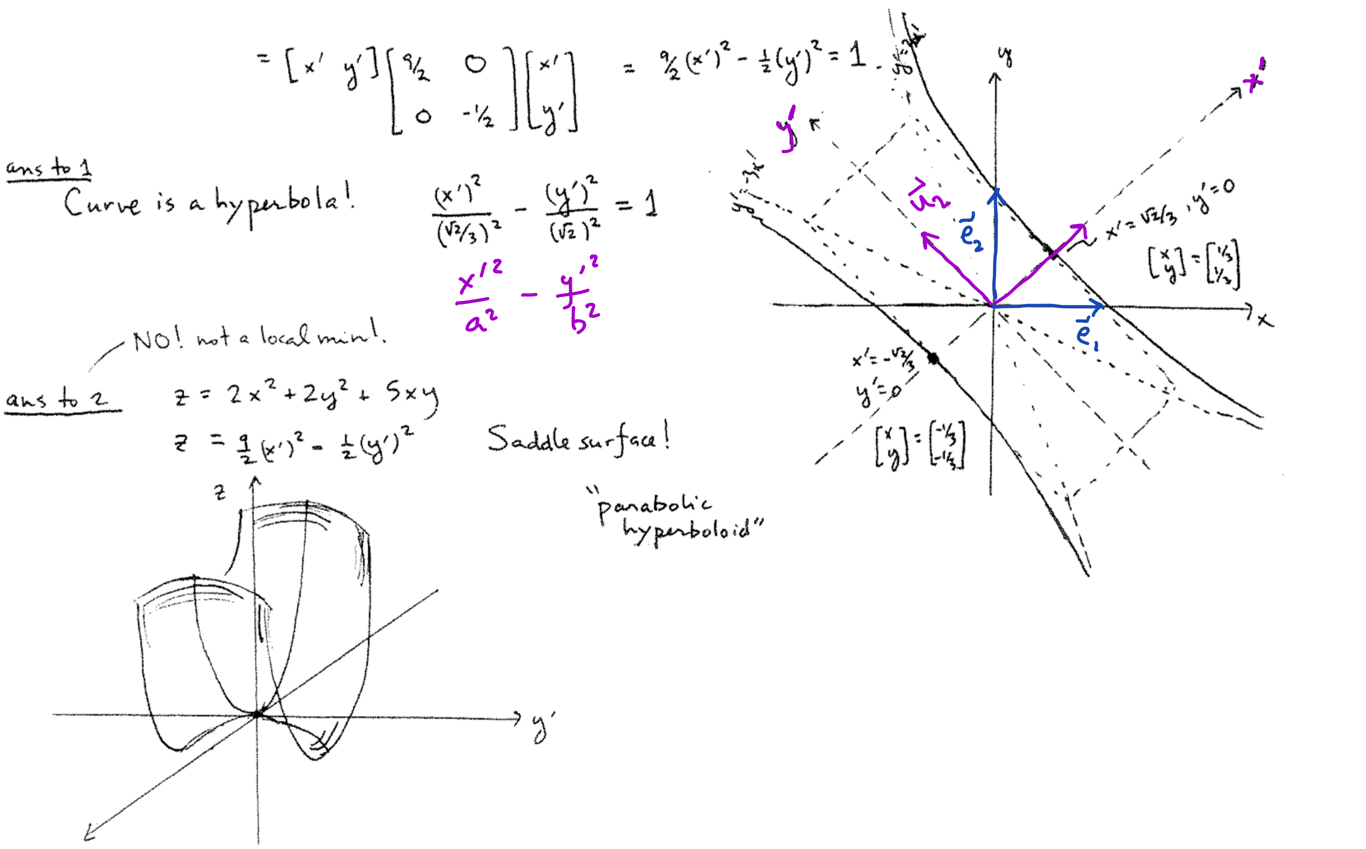
$$\frac{(x')^2}{\left(\sqrt{\frac{2}{3}}\right)^2} - \frac{(y')^2}{(\sqrt{2})^2} = 1.$$

Answer to 1b) No! $f(x, y) = 2x^2 + 2y^2 + 5xy$ does not have a local min or max at $(0, 0)$. The origin is a saddle point, because in the rotated coordinate system

$$f(x', y') = \frac{9}{2}(x')^2 - \frac{1}{2}(y')^2.$$

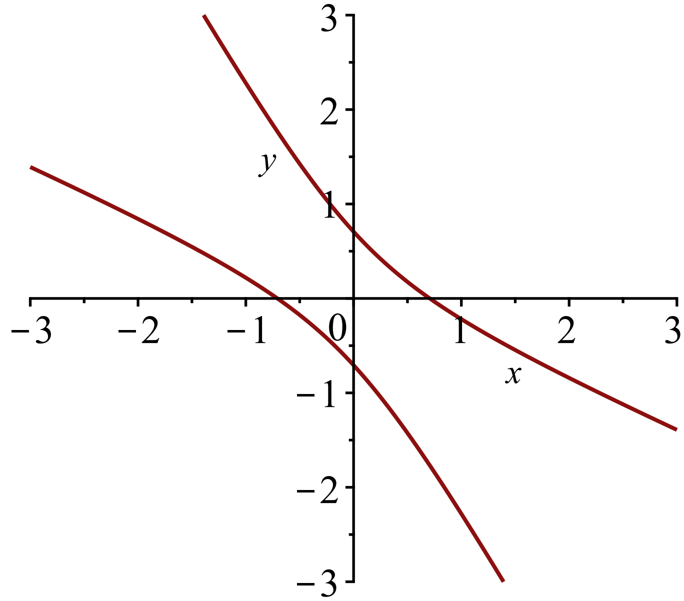
Old pictures from when I could still sketch well:

$$\{\vec{u}_1, \vec{u}_2\} = \left\{ \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \right\}$$

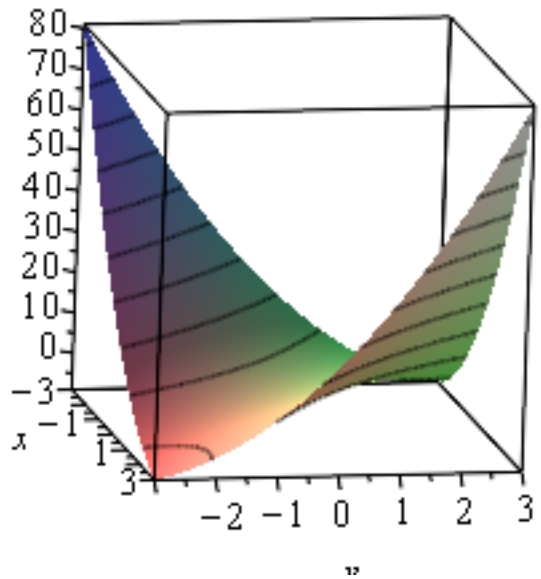


Maple verification: To be continued ....

> $with(plots):$
$implicitplot(2\cdot x^2 + 2\cdot y^2 + 5\cdot x\cdot y = 1, x = -3..3, y = -3..3, grid = [200, 200]);$



> $plot3d(2\cdot x^2 + 2\cdot y^2 + 5\cdot x\cdot y, x = -3..3, y = -3..3);$



Wed Apr 18

- 7.1-7.2 Diagonalizing quadratic forms and surfaces and curves defined implicitly with quadratic equations, via the spectral theorem continued; with proof of spectral theorem appended.

Announcements:
- Guest lecture on principal componant analysis (§7.5) Friday (Prof. Tom Alberts)
- prep notes

Warm-up Exercise: Compute

$$\overset{A}{\begin{bmatrix} 1 & -2 \\ 2 & 0 \end{bmatrix}} \overset{B}{\begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 1 \end{bmatrix}} = \begin{bmatrix} 1 & -4 & 1 \\ 2 & 0 & 6 \end{bmatrix}$$

"inner product"

then compute $\sum_{j=1}^{2} [col_j(A)][row_j(B)]$

$$= \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 3 \\ 2 & 0 & 6 \end{bmatrix} + \begin{bmatrix} 0 & -4 & -2 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -4 & 1 \\ 2 & 0 & 6 \end{bmatrix}$$

"outer product"

Spectral Theorem Let A be an $n \times n$ symmetric matrix. Then all of the eigenvalues of A are real, and there exists an orthonormal eigenbasis $B = \{\underline{\boldsymbol{u}}_1, \underline{\boldsymbol{u}}_2, \dots \underline{\boldsymbol{u}}_n\}$ consisting of eigenvectors for A. Eigenspaces with different eigenvalues are automatically orthogonal to each other. If any eigenspace has dimension greater than 1, its orthonormal basis may be constructed via $Gram - Schmidt$. (Proof of spectral theorem at end of today's notes.)

Diagonalization of quadratic forms: Let

$$Q(\underline{x}) = \sum_{i,j=1}^{n} a_{ij} x_i x_j = \underline{x}^T A \underline{x}$$

for a symmetric matrix A, with real entries. A symmetric $\Rightarrow$ by the spectral theorem there exists an orthonormal eigenbasis $B = \{\underline{\boldsymbol{u}}_1, \underline{\boldsymbol{u}}_2, \dots \underline{\boldsymbol{u}}_n\}$.

For the corresponding orthogonal matrix

$$P = [\underline{\boldsymbol{u}}_1 | \underline{\boldsymbol{u}}_2 | \cdots | \underline{\boldsymbol{u}}_n]$$

$$D = P^T A P$$

$AP = PD$

$P^{-1}AP = D$

$P^{-1} = P^T$

where D is the diagonal matrix of eigenvalues corresponding to the eigenvectors in P. And we have

$$\underline{x} = P\underline{y}$$

where $\underline{y} = [\underline{x}]_B$ and $P = P_{E \leftarrow B}$. Thus

$\vec{x}^T = \vec{y}^T P^T$

$$Q(\underline{x}) = \underline{x}^T A \underline{x}$$

$$= \underline{y}^T P^T A P \underline{y} = \underline{y}^T D \underline{y}$$

$$= \sum_{i=1}^{n} \lambda_i y_i^2 .$$

So by the orthogonal change of variables all cross terms have been removed. Applications include conic curves, quartic surfaces, multivariable second derivative test, singular value decomposition theorem, and more.

Material we need for Prof. Alberts' guest lecture Friday on Principal Component Analysis. (The text discusses most of this background material in 7.1, 7.2)

Definition: The quadratic form $Q(\underline{x}) = \sum_{i,j=1}^{n} a_{ij} x_i x_j = \underline{x}^T A \underline{x}$ (for A a symmetric matrix) is called *positive definite* if

$$Q(\underline{x}) > 0 \quad \text{for all } \underline{x} \neq \underline{0}.$$

From the previous page, we see that this is the same as saying that all of the eigenvalues of A are positive.

Theorem: The "outer product" way of computing the matrix product $A\,B$. (Section 2.4 topic on partitioned matrices that we skipped....our usual way is with dot product or rows of A with columns of B, aka an "inner product").

(1) first, notice that the product of an $m \times 1$ column vector with a $1 \times n$ row vector is an $m \times n$ matrix:

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \begin{bmatrix} b_1 & b_2 \end{bmatrix} = \begin{bmatrix} a_1 b_1 & a_1 b_2 \\ a_2 b_1 & a_2 b_2 \\ a_3 b_1 & a_3 b_2 \end{bmatrix}.$$

(1) Let $A_{m \times p}$ and $B_{p \times n}$. Express A in terms of its columns, and B in terms of its rows:

$$A = \begin{bmatrix} | & | & & | \\ \underline{a}_1 & \underline{a}_2 & \cdots & \underline{a}_p \\ | & | & & | \\ | & | & & | \end{bmatrix} \qquad B = \begin{bmatrix} ---\underline{b}_1--- \\ ---\underline{b}_2--- \\ \vdots \\ ---\underline{b}_p--- \end{bmatrix}.$$

Then

$$A\,B = \sum_{j=1}^{p} \underline{a}_j \, \underline{b}_j .$$

We can illustrate the general proof by considering the example in which A and B are each 3 × 3: Look column by column in the output of each expression to verify the identity, using the the linear combination form of matrix times vector, for $A\,B$:

$A\,\mathrm{col}_1(\vec{B})$

$= b_{11}\,\mathrm{col}_1(A) + b_{21}\,\mathrm{col}_2(A) + b_{31}\,\mathrm{col}_3(A)$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} \\ a_{12} \\ a_{13} \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} & b_{13} \end{bmatrix} + \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix}\begin{bmatrix} b_{21} & b_{22} & b_{23} \end{bmatrix} + \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix}\begin{bmatrix} b_{31} & b_{32} & b_{33} \end{bmatrix}.$$

actual proof with formula

via dot product: $\text{entry}_{k\ell}\,AB = \text{row}_k(A)\cdot\text{col}_\ell(B) = \sum_{j=1}^{p} a_{kj}\,b_{j\ell}$

Same!

via outer product: $\text{entry}_{k\ell}\begin{bmatrix}\vec{a}_j\end{bmatrix}\begin{bmatrix}-\vec{b}_j-\end{bmatrix} = a_{kj}\,b_{j\ell}$

$\sum_{j=1}^{p}\text{entry}_{k\ell}\begin{bmatrix}\vec{a}_j\end{bmatrix}\begin{bmatrix}-\vec{b}_j-\end{bmatrix} = \sum_{j=1}^{p} a_{kj}\,b_{j\ell}$

$\quad = \text{entry}_{k\ell}\sum_{j=1}^{p}\begin{bmatrix}\vec{a}_j\end{bmatrix}\begin{bmatrix}-\vec{b}_j-\end{bmatrix}$

Spectral decomposition for symmetric matrices. Let $A_{n \times n}$ be symmetric (and positive definite, for the applications Prof. Alberts will talk about on Friday). Order the eigenvalues as

$$\lambda_1 \geq \lambda_2 \geq \ldots \lambda_n > 0$$

and let

$$\{\underline{\boldsymbol{u}}_1, \underline{\boldsymbol{u}}_2, \cdots, \underline{\boldsymbol{u}}_n\}$$

be a corresponding orthonormal eigenbasis of $\mathbb{R}^n$. Let P be the orthogonal matrix

$$P = [\underline{\boldsymbol{u}}_1, \underline{\boldsymbol{u}}_2, \ldots \underline{\boldsymbol{u}}_n]$$

with

$$A\,P = P\,D$$

where D is the diagonal matrix with diagonal entries $\lambda_1 \geq \lambda_2 \geq \ldots \lambda_n > 0$.

$D = P^T A P$

$AP = PD$

$APP^T = PDP^T$

$A = PDP^T$

$(P^{-1} = P^T)$

Then

$$A = P\,D\,P^T$$

$$= \begin{bmatrix} | & | & & | \\ \underline{\boldsymbol{u}}_1 & \underline{\boldsymbol{u}}_2 & \ldots & \underline{\boldsymbol{u}}_n \\ | & | & & | \\ | & | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ : & \ldots & \ldots & 0 \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix} \begin{bmatrix} ---\underline{\boldsymbol{u}}_1^T--- \\ ---\underline{\boldsymbol{u}}_2^T--- \\ : \\ ---\underline{\boldsymbol{u}}_n^T--- \end{bmatrix}$$

$$= \begin{bmatrix} | & | & & | \\ \lambda_1\underline{\boldsymbol{u}}_1 & \lambda_2\underline{\boldsymbol{u}}_2 & \ldots & \lambda_n\underline{\boldsymbol{u}}_n \\ | & | & & | \\ | & | & & | \end{bmatrix} \begin{bmatrix} ---\underline{\boldsymbol{u}}_1^T--- \\ ---\underline{\boldsymbol{u}}_2^T--- \\ : \\ ---\underline{\boldsymbol{u}}_n^T--- \end{bmatrix}$$

$$A = \lambda_1\,\underline{\boldsymbol{u}}_1\underline{\boldsymbol{u}}_1^T + \lambda_2\,\underline{\boldsymbol{u}}_2\,\underline{\boldsymbol{u}}_2^T + \ldots + \lambda_n\underline{\boldsymbol{u}}_n\,\underline{\boldsymbol{u}}_n^T$$

"Principal component analysis" makes use of the fact that if only a few of the eigenvalues of A are large and the rest are near zero, then the corresponding leading terms in the expression above are a good approximation for the matrix A.

Remark: There's slick way to see this spectral decomposition matrix identity that doesn't use the outer product but uses our work on projection instead:

$$\underline{\boldsymbol{x}} = (\underline{\boldsymbol{x}} \cdot \underline{\boldsymbol{u}}_1)\underline{\boldsymbol{u}}_1 + (\underline{\boldsymbol{x}} \cdot \underline{\boldsymbol{u}}_2)\underline{\boldsymbol{u}}_2 + \cdots (\underline{\boldsymbol{x}} \cdot \underline{\boldsymbol{u}}_n)\underline{\boldsymbol{u}}_n$$

$$\Rightarrow A\underline{\boldsymbol{x}} = (\underline{\boldsymbol{x}} \cdot \underline{\boldsymbol{u}}_1)\lambda_1\underline{\boldsymbol{u}}_1 + (\underline{\boldsymbol{x}} \cdot \underline{\boldsymbol{u}}_2)\lambda_2\underline{\boldsymbol{u}}_2 + \quad (\underline{\boldsymbol{x}} \cdot \underline{\boldsymbol{u}}_n)\lambda_n\underline{\boldsymbol{u}}_n$$

$$= \lambda_1 \underline{\boldsymbol{u}}_1 \left(\underline{\boldsymbol{u}}_1^T \underline{\boldsymbol{x}}\right) + \lambda_2\underline{\boldsymbol{u}}_2\left(\underline{\boldsymbol{u}}_2^T\underline{\boldsymbol{x}}\right) + \quad + \lambda_n\underline{\boldsymbol{u}}_n\left(\underline{\boldsymbol{u}}_n^T\underline{\boldsymbol{x}}\right)$$

$$A\underline{\boldsymbol{x}} = \left[\, \lambda_1 \underline{\boldsymbol{u}}_1 \underline{\boldsymbol{u}}_1^T + \lambda_2 \underline{\boldsymbol{u}}_2 \underline{\boldsymbol{u}}_2^T + \ldots + \lambda_n \underline{\boldsymbol{u}}_n \underline{\boldsymbol{u}}_n^T \right] \underline{\boldsymbol{x}}\,.$$

Since this is true for all $\underline{\boldsymbol{x}}$, (in particular for the standard basis vectors, which lets us recover the columns of A) we deduce

$$A = \lambda_1 \underline{\boldsymbol{u}}_1 \underline{\boldsymbol{u}}_1^T + \lambda_2 \underline{\boldsymbol{u}}_2 \underline{\boldsymbol{u}}_2^T + \ldots + \lambda_n \underline{\boldsymbol{u}}_n \underline{\boldsymbol{u}}_n^T\,.$$

Testing spectral decomposition in a small example:

$$A = \begin{bmatrix} 2 & \frac{5}{2} \\ \frac{5}{2} & 2 \end{bmatrix}$$

$$E_{\lambda = \frac{9}{2}} = span\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\} \quad E_{\lambda = -\frac{1}{2}} = span\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$$

$$\underline{\boldsymbol{u}}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \underline{\boldsymbol{u}}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\lambda_1 \underline{\boldsymbol{u}}_1 \underline{\boldsymbol{u}}_1^T + \lambda_2 \underline{\boldsymbol{u}}_2 \underline{\boldsymbol{u}}_2^T = \frac{9}{2} \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \end{bmatrix} - \frac{1}{2} \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \end{bmatrix}$$

$$= \frac{9}{4}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - \frac{1}{4}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & \frac{5}{2} \\ \frac{5}{2} & 2 \end{bmatrix} !!!!$$

Example Identify and sketch the surface defined implicitly by

$$x_1^2 + x_2^2 - 2x_3^2 - 2x_1 x_2 - 4x_1 x_3 - 4x_2 x_3 = 8.$$

Exercise 1) Find the symmetric matrix so that

$$x_1^2 + x_2^2 - 2x_3^2 - 2x_1 x_2 - 4x_1 x_3 - 4x_2 x_3 = \underline{\boldsymbol{x}}^T A\, \boldsymbol{x}.$$

Recall that

$$\underline{\boldsymbol{x}}^T A\, \boldsymbol{x} = \sum_{i,j=1}^{n} a_{ij} x_i x_j .$$

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1 & -1 & -2 \\ -1 & 1 & -2 \\ -2 & -2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

If we found the matrix correctly technology tells us that

$$E_{\lambda=-2} = span\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\},\ E_{\lambda=2} = span\left\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right\},\ E_{\lambda=4} = span\left\{\begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}\right\}.$$

(positively oriented in this order)

$$x_1^2 + x_2^2 - 2x_3^2 - 2x_1x_2 - 4x_1x_3 - 4x_2x_3 = 8$$

$$\underline{\boldsymbol{x}}^T A\, \underline{\boldsymbol{x}} = 8$$

For

$$P = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{bmatrix}$$

$$P^T A\, P = D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$\underline{\boldsymbol{x}}^T A\, \underline{\boldsymbol{x}} = 8$$

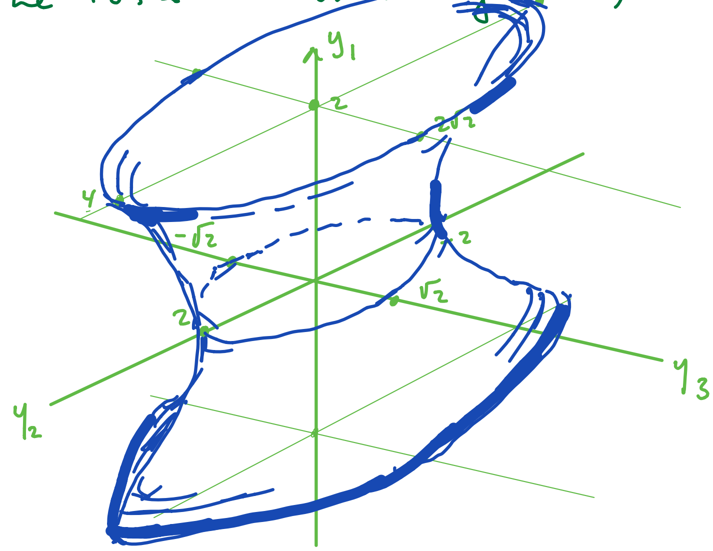
$$\underline{\boldsymbol{y}}^T P^T A\, P\, \underline{\boldsymbol{y}} = 8$$

$$\underline{\boldsymbol{y}}^T \mathrm{D}\, \underline{\boldsymbol{y}} = 8$$

$$-2y_1^2 + 2y_2^2 + 4y_3^2 = 8. \qquad 2y_2^2 + 4y_3^2 = 8 + 2y_1^2$$

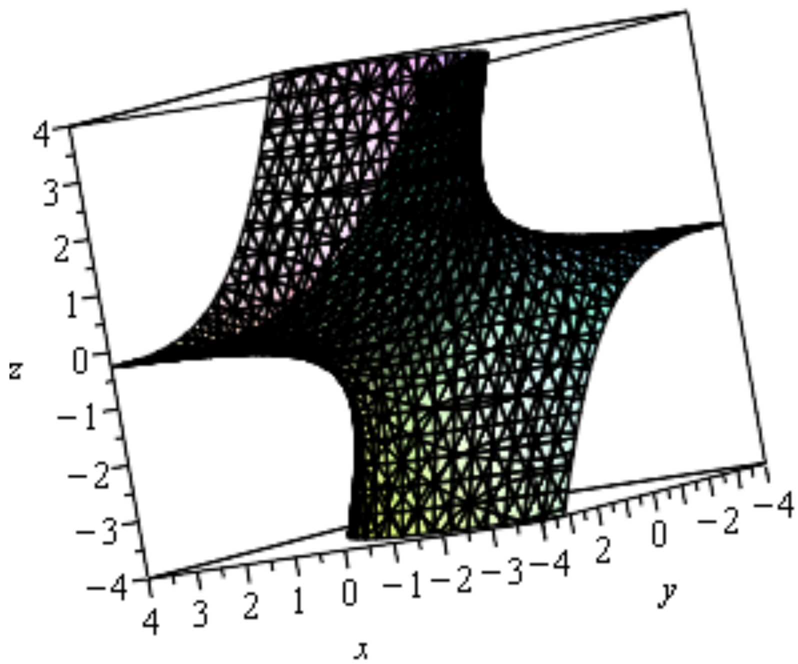
We can try to sketch this in the rotated coordinate system. It's an elliptic hyperboloid of one sheet. :-)

(Notice that when all is said & done, we only needed the eigenvalues of A to sketch the quadric surface with respect to the rotated coord. system.)



> $with(plots):$

$implicitplot3d\left(x^2 + y^2 - 2\,z^2 - 2\,x\cdot y - 4\cdot x\cdot z - 4\cdot y\cdot z = 8, x = -4..4, y = -4..4, z = -4..4, grid = [20, 20, 20]\right);$




eigenvalues{{1,-1,-2},{-1,1,-2},{-2,-2,2}}

Browse Examples   Surprise Me

Input:

eigenvalues $\begin{pmatrix} 1 & -1 & -2 \\ -1 & 1 & -2 \\ -2 & -2 & 2 \end{pmatrix}$

Open code

Results: Step-by-step solution

$\lambda_1 = 4$

$\lambda_2 = -2$

$\lambda_3 = 2$

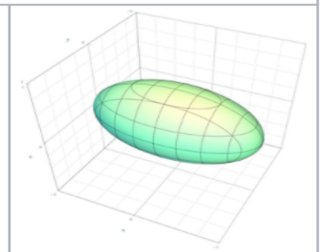
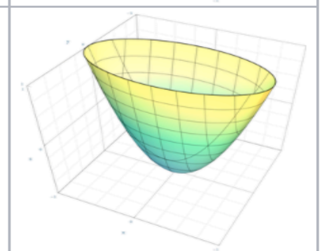
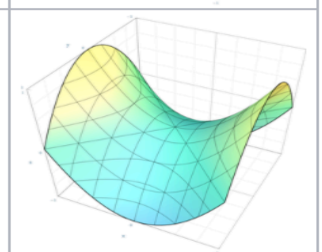
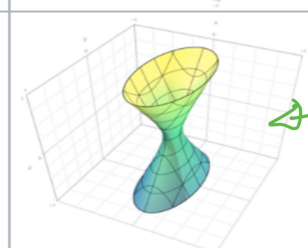
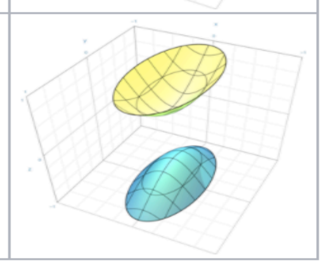
Corresponding eigenvectors: Step-by-step solution

$v_1 = (-1, -1, 2)$

$v_2 = (1, 1, 1)$

$v_3 = (-1, 1, 0)$

from Wikipedia, "quadric surfaces". There is also a Wikipedia page on conic sections.

Non-degenerate real quadric surfaces		
Ellipsoid	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$	
Elliptic paraboloid	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - z = 0$	
Hyperbolic paraboloid	$\frac{x^2}{a^2} - \frac{y^2}{b^2} - z = 0$	
Elliptic hyperboloid of one sheet	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$	
Elliptic hyperboloid of two sheets	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$	

Spectral Theorem Let $A_{n\times n}$ be a real, symmetric matrix.
Then $\exists$ an orthonormal $\mathbb{R}^n$ basis made of eigenvectors of A, $\mathcal{B}=\{\vec{u}_1,\vec{u}_2,\ldots\vec{u}_n\}$ $A\vec{u}_j=\lambda_j\vec{u}_j$
Thus for $S=[\vec{u}_1|\vec{u}_2|\cdots\vec{u}_n]$,

$$S^TAS=\begin{bmatrix}\lambda_1 & & O\\ & \lambda_2 & \\ O & \ddots & \lambda_n\end{bmatrix}$$ is diagonal.

proof

(1) On Monday we showed that if $\lambda_1\neq\lambda_2$ are real eigenvalues of A, with eigenvectors $\vec{v}_1,\vec{v}_2\neq\vec{0}$

$$A\vec{v}_1=\lambda_1\vec{v}_1 \qquad A\vec{v}_2=\lambda_2\vec{v}_2$$

Then $\vec{v}_1\perp\vec{v}_2$

proof $v_2^TA\vec{v}_1=v_2^T(\lambda_1\vec{v}_1)=\lambda_1\vec{v}_2\cdot\vec{v}_1$
$\quad\| $
$(v_2^TA^T)\vec{v}_1=(A\vec{v}_2)^T\vec{v}_1$
$=\lambda_2\vec{v}_2^T\vec{v}_1$
$=\lambda_2\vec{v}_2\cdot\vec{v}_1$
So $\vec{v}_2\cdot\vec{v}_1=0$ ∎

We also showed that for $A_{2\times2}$ symmetric, either A is already diagonal (a multiple of I, in fact), or A has 2 distinct real eigenvalues $\Rightarrow A$ diagonalizable. By (1) the eigenvectors are $\perp$, so normalize to get orthonormal eigenbasis.

(2) All eigenvalues of A are real: Let $\lambda=a+bi$ be any root of ~~$f_A(\lambda)$~~ $p(\lambda)$, and let $\vec{u}+i\vec{v}$ be a corresponding non-zero eigenvector.

$$A(\vec{u}+i\vec{v})=(a+ib)(\vec{u}+i\vec{v})$$

take conjugates: $A(\vec{u}-i\vec{v})=(a-ib)(\vec{u}-i\vec{v})$.

Now consider

$$(\vec{u}-i\vec{v})^TA(\vec{u}+i\vec{v})=(\vec{u}-i\vec{v})^T(a+ib)(\vec{u}+i\vec{v})=(a+ib)\left[(\vec{u}-i\vec{v})^T(\vec{u}+i\vec{v})\right]$$
$$=(a+ib)\left[(\vec{u}-i\vec{v})\cdot(\vec{u}+i\vec{v})\right]$$
$$=(a+ib)\left(\|\vec{u}\|^2+\|\vec{v}\|^2\right)$$

$$\| $$

$$\left((\vec{u}-i\vec{v})^TA^T\right)(\vec{u}+i\vec{v})$$

$$\| $$

$$\left[A(\vec{u}-i\vec{v})\right]^T(\vec{u}+i\vec{v})=(a-ib)(\vec{u}-i\vec{v})^T(\vec{u}+i\vec{v})$$
$$=(a-ib)\left(\|\vec{u}\|^2+\|\vec{v}\|^2\right)$$

$\Rightarrow b=0!$ ∎

thus ~~$f_A(\lambda)$~~ $p(\lambda)$ factors completely over $\mathbb{R}$.

- If it has n distinct roots, then alg mult = geom mult = 1, all evects w different evals are $\perp$ by (1), and normalize to ~~get~~ get orthonormal eigenbasis
- Otherwise it's a little harder: (In practice, if λ_i has alg & geom mult $k_i>1$ just Gram-~~Sch~~ Schmidt its eigenbasis)

(3) General proof, by induction:
Spectral Theorem true for $n=1$ (1×1 matrices are diagonal)
$n=2$ (we checked yesterday).

Inductive step:

- Assume all $(n-1)\times(n-1)$ symmetric matrices are diagonalizable with an orthogonal matrix (with eigenbasis columns).

- Now let $A_{n\times n}$ symmetric

Let λ_1 be any root of ~~$f_A(\lambda)$~~ $p(\lambda)$. λ_1 is real by (2).
Let $\vec{u}_1$ be a unit eigenvector

$$A\vec{u}_1 = \lambda_1\vec{u}_1, \quad \|\vec{u}_1\| = 1.$$

Complete to an orthonormal basis ← probably not eigenvectors

$$\mathcal{B}_0 = \{\vec{u}_1, \underbrace{\vec{v}_2, \vec{v}_3, \dots \vec{v}_n}\} \text{ of } \mathbb{R}^n$$

$$S_0 = \left[\vec{u}_1 \middle| \vec{v}_2 \middle| \dots \vec{v}_n\right] \qquad S_0^T S_0 = I$$

$S_0^T A S_0$ <u>is</u> symmetric (take its transpose!)

1st column <u>is</u> $\begin{bmatrix}\lambda_1\\0\\\vdots\\0\end{bmatrix}$

so $$S_0^T A S_0 = \left[\begin{array}{c|c}\lambda_1 & 0 \\ \hline 0 & \\ 0 & B \\ \vdots & \\ 0 & \end{array}\right]$$

$B_{(n-1)\times(n-1)}$ is symmetric, so by induction hypothesis $\exists$ S_1 orthog, with $S_1^T B S_1 = \begin{bmatrix}\lambda_2 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{bmatrix}$ (λ_i's need not be distinct!)

Thus $$\left[\begin{array}{c|c}1 & 0\\ \hline 0 & S_1^T\end{array}\right]\underbrace{\left[\begin{array}{c|c}\lambda_1 & 0\\ \hline 0 & B\end{array}\right]}_{S_0^T A S_0}\left[\begin{array}{c|c}1 & 0\\ \hline 0 & S_1\end{array}\right] = \begin{bmatrix}\lambda_1 & & & 0\\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n\end{bmatrix}$$

so $S^T A S = D$

$$S = S_0 \left[\begin{array}{c|c}1 & 0\\ \hline 0 & S_1\end{array}\right] \quad \text{orthog (product of orthog)}$$

∎