Tues Apr 17

- Symmetric matrices and the spectral theorem, 7.1-7.2

Chapter 7: spectral theorem & applications

Announcements:

Warm-up Exercise: Find eigenvalues & eigenspace bases for

$$A = \begin{bmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 0 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 0 \end{bmatrix} = A$$

$$|A-\lambda I| = (\lambda-2)(\lambda+1).$$

$$A - \lambda I = \begin{bmatrix} 1-\lambda & \sqrt{2} \\ \sqrt{2} & -\lambda \end{bmatrix}$$

$$|A-\lambda I| = \lambda^2-\lambda-2 = (\lambda-2)(\lambda+1)$$

$E_{\lambda=2}$:
$$\left[\begin{array}{cc|c} -1 & \sqrt{2} & 0 \\ \sqrt{2} & -2 & 0 \end{array}\right]$$

$Row_2 = -\sqrt{2}\, Row_1$

$$\left[\begin{array}{cc|c} -1 & \sqrt{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$E_{\lambda=2} = \text{span}\left\{ \begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix} \right\}.$$

$E_{\lambda=-1}$
$$\left[\begin{array}{cc|c} 2 & \sqrt{2} & 0 \\ \sqrt{2} & 1 & 0 \end{array}\right]$$

$Row_1 = \sqrt{2}\, Row_2$

$$E_{\lambda=-1} = \text{span}\left\{ \begin{bmatrix} -\sqrt{2} \\ 2 \end{bmatrix} \right\} = \text{span}\left\{ \begin{bmatrix} 1 \\ -\sqrt{2} \end{bmatrix} \right\}$$

$$\begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -\sqrt{2} \end{bmatrix} = 0 \ !$$

Recall that the transpose operation swaps rows with columns, and vise verse. These properties arose from the actual definition for A^T, which was

$$entry_{i\,j}A^T = entry_{j\,i}A.$$

The $i\,j$ and $j\,i$ locations on a matrix are reflections across the diagonal of each other. (This is the matrix version of the $\mathbb{R}^2$ reflection across the line $x_2 = x_1$ that we've encountered several times in this course.) See how this plays out for the matrix A below, by finding the transpose three ways: Turning rows into columns; turning columns into rows; reflecting across the diagonal.

$$A = \begin{bmatrix} 1 & 2 & 7 \\ -1 & 3 & 2 \\ 9 & 4 & 2 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & -1 & 9 \\ 2 & 3 & 4 \\ 7 & 2 & 2 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & -1 & 9 \\ 2 & 3 & 4 \\ 7 & 2 & 2 \end{bmatrix}$$

③ reflect across diag.

$$A^T = \begin{bmatrix} 1 & -1 & 9 \\ 2 & 3 & 4 \\ 7 & 2 & 2 \end{bmatrix}$$

Def A square matrix is *symmetric* if and only if $A^T = A$.

Exercise 1 Which of the following matrices is symmetric, and which is not?

1a)

$$B := \begin{bmatrix} 4 & 2 & 1 \\ 2 & 0 & -2 \\ 1 & -2 & 7 \end{bmatrix}$$

B is symmetric

1b)

$$C := \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -2 \\ 2 & -2 & 3 \end{bmatrix}$$

NO

The *Spectral Theorem* asserts that all $n \times n$ symmetric matrices A (with real number entries) are diagonalizable, with n linearly independent real eigenvectors and associated eigenvalues. Furthermore, eigenvectors with different eigenvalues are automatically orthogonal. (For eigenspaces with dimension greater than one, one can use Gram Schmidt to create orthonormal bases). Thus, the eigenvector basis of $\mathbb{R}^n$ can be chosen to be orthonormal. In otherwords, we may express

$$AP = PD$$

$$\cancel{A = P^{-1} D P = P^T D P}$$

$$A = PDP^{-1} = PDP^T$$

$$D = P^{-1}AP = P^TAP$$

where P is an orthogonal matrix which can also be interpreted as a change of basis matrix. Let's see how this plays out in an example. This will forshadow all of sections 7.1-7.2. You'll notice that we're using major concepts and ideas from throughout the course, which is not a bad way to be reviewing course material at this point of the semester.

Example

1 Consider the curve in $\mathbb{R}^2$ defined implicitly as the solution set to the equation

$$2x^2 + 2y^2 + 5xy = 1.$$

Can you identify the curve as a conic section? Can you graph it? Note the xy term!

2 Does the function $f(x, y) = 2x^2 + 2y^2 + 5xy$ have a local maximum or local minimum at $(x, y) = (0, 0)$? Note, the gradient

$$\nabla f = [f_x, f_y] = [4x + 5y, 4y + 5x] = [0, 0] \text{ at the point } (0, 0),$$

so the origin is at least a candidate for a local max or min.

Exercise 1a. Check that can rewrite the quadratic expression as ✓

$$2x^2 + \boxed{2y^2} + 5xy = [\,x \;\; y\,] \begin{bmatrix} 2 & \frac{5}{2} \\ \frac{5}{2} & \boxed{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

$$= [x, y] \begin{bmatrix} 2x + \frac{5}{2}y \\ \frac{5}{2}x + \boxed{2}y \end{bmatrix}$$

$$= x\left(2x + \frac{5}{2}y\right) + y\left(\frac{5}{2}x + 2y\right)$$

$$2x^2 + \frac{5}{2}xy + \frac{5}{2}yx + 2y^2$$

Note, in general, if $\underline{\boldsymbol{v}}, \underline{\boldsymbol{w}} \in \mathbb{R}^n$ and if A is an $n \times n$ matrix then

$$\underline{\boldsymbol{v}}^T_{1 \times \cancel{n}} A_{\cancel{n} \times \cancel{n}} \underline{\boldsymbol{w}}_{\cancel{n} \times 1} = [\vec{v}^T A \vec{w}]_{1\times 1} \quad \text{scalar}$$

is a 1×1 matrix, i.e. a scalar. And its value is

$$\underline{\boldsymbol{v}}^T A\, \underline{\boldsymbol{w}} = \sum_{i=1}^{n} v_i \left(entry_i \left(A\, \underline{\boldsymbol{w}} \right) \right) = \sum_{i=1}^{n} v_i \left(\sum_{j=1}^{n} a_{ij} w_j \right) = \sum_{i,j=1}^{n} a_{ij} v_i w_j .$$

So given a quadratic expression ("quadratic form") in any number of variables $(x_1, x_2, \ x_n)$ one can rewrite the quadratic form as

$$\underline{\boldsymbol{x}}^T A\, \underline{\boldsymbol{x}}$$

and one can choose to make A a symmetric matrix, as we did in our specific example. by splitting cross terms symmetrically.

$$\vec{x}^T A \vec{x} = \sum_{i,j=1}^{n} a_{ij} x_i x_j$$

a_{11} for x_1^2
a_{22} for x_2^2
a_{33} for x_3^2

e.g. $x_1^2 + 4x_2^2 + 2x_3^2 + 2x_1x_2 - 1x_1x_3 + 5x_2x_3$

"quadratic form"

$$= \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1/2 \\ 1 & 4 & 5/2 \\ -1/2 & 5/2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$1\, x_1x_2 + 1\, x_2x_1$

$-\frac{1}{2} x_1x_3 - \frac{1}{2} x_3x_1$

Exercise 1a Find the eigenvalues and eigenvectors for the matrix we're using to express our quadratic expression.

$$2x^2 + 2y^2 + 5xy = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 2 & \frac{5}{2} \\ \frac{5}{2} & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Check eigendata

$$\begin{bmatrix} 2 & 5/2 \\ 5/2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \checkmark \qquad A\vec{v} = \lambda \vec{v}$$

$$\begin{bmatrix} 2 & 5/2 \\ 5/2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 9/2 \\ 9/2 \end{bmatrix} = \frac{9}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \checkmark \qquad \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0$$

Solution: $E_{\lambda = -\frac{1}{2}} = span\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$

$E_{\lambda = \frac{9}{2}} = span\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}.$

Was it an accident that the two eigenvectors were orthogonal? No. Here's why that will always be true as long as the eigenvalues are different, for any symmetric matrix of arbitrary size: Let A be symmetric, and let

$$A\,\underline{v} = \lambda_1 \underline{v} \qquad A\,\underline{w} = \lambda_2\,\underline{w}$$

with $\lambda_1 \neq \lambda_2$. Because $A^T = A$, we claim that

$$\underline{w} \cdot A\,\underline{v} = A\,\underline{w} \cdot \underline{v}.$$

(in general $\vec{w} \cdot A\vec{v} = A^T\vec{w} \cdot \vec{v}$)

One way to see this is by noting

$$\boxed{\underline{w} \cdot A\,\underline{v}} = \underline{w}^T A\,\underline{v}\ .$$

• $(AB)^T = B^T A^T$

Since the result of this operation is a scalar, it equals its transpose:

$$\boxed{\underbrace{\left(\underline{w}^T A\,\underline{v}\right)}_{[\ ]_{1\times 1}}} = \left(\underline{w}^T A\,\underline{v}\right)^T = \underline{v}^T A^T \underline{w} = \underline{v}^T A\,\underline{w} = \boxed{\underline{v} \cdot A\,\underline{w}\,.}$$

$A^T = A$

But

$$\underline{w} \cdot A\,\underline{v} = \underline{w} \cdot \lambda_1\,\underline{v} = \lambda_1\,\underline{v} \cdot \underline{w}.$$

$$A\,\underline{w} \cdot \underline{v} = \lambda_2\,\underline{w} \cdot v.$$

$\lambda_1\, \vec{v}\cdot\vec{w} = \lambda_2\, \vec{v}\cdot\vec{w}$

$\lambda_1 \neq \lambda_2 \Rightarrow \vec{v}\cdot\vec{w} = \vec{0}.$

So, since $\lambda_1 \neq \lambda_2$ is must be that $\underline{v} \cdot \underline{w} = 0$!

* And a special fact for 2×2 symmetric matrices and eigenvectors in $\mathbb{R}^2$: If $A\,\underline{v} = \lambda\,\underline{v}$ for $\underline{v} \neq 0$ let $\underline{w} \perp \underline{v}$. Then $\underline{w}$ is automatically an eigenvector:

$$\underline{w} \cdot A\,\underline{v} = \underline{w} \cdot (\lambda\,\underline{v}) = \lambda\,\underline{w} \cdot \underline{v} = 0.$$

true but don't need it,

So

$$0 = \underline{w} \cdot A\,\underline{v} = \underline{v} \cdot A\,\underline{w} \Rightarrow \underline{v} \perp A\,\underline{w} \Rightarrow A\,\underline{w} \in span\{\underline{w}\}$$

because we're in $\mathbb{R}^2$. So $\underline{w}$ is also an eigenvector, automatically.

Theorem: Spectral theorem for 2×2 symmetric matrices

Let $A = \begin{bmatrix} a & c \\ c & b \end{bmatrix}$.

$$A - \lambda I = \begin{bmatrix} a-\lambda & c \\ c & b-\lambda \end{bmatrix}$$

$$|A - \lambda I| = (\lambda - a)(\lambda - b) - c^2$$
$$(a-\lambda)(b-\lambda)$$
$$= \lambda^2 - (a+b)\lambda + ab - c^2$$
$$\lambda = \frac{(a+b) \pm \sqrt{(a+b)^2 - 4(ab - c^2)}}{2}$$
$$= \frac{(a+b) \pm \sqrt{a^2 + 2ab + b^2 - 4ab + 4c^2}}{2}$$

$$\lambda = \frac{a+b \pm \sqrt{(a-b)^2+4c^2}}{2}$$

real if $(a-b)^2+4c^2>0$ then $\lambda_1 \neq \lambda_2$, $\vec{v}_1 \perp \vec{v}_2$

So $\left\{ \frac{\vec{v}_1}{\|\vec{v}_1\|}, \frac{\vec{v}_2}{\|\vec{v}_2\|} \right\}$ o.n. basis for $\mathbb{R}^2$.

Else $(a-b)^2+4c^2=0$

$\Rightarrow a=b,\ c=0$

$A=\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ & $\{\vec{e}_1,\vec{e}_2\}$ o.n. eigenbasis

Continuing ...

$$2x^2+2y^2+5xy = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 2 & \frac{5}{2} \\ \frac{5}{2} & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

and for

$$A = \begin{bmatrix} 2 & \frac{5}{2} \\ \frac{5}{2} & 2 \end{bmatrix}; \qquad E_{\lambda=-\frac{1}{2}} = span\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}, \ E_{\lambda=\frac{9}{2}} = span\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}.$$

This suggests creating an orthonormal eigenbasis! And we'll order the eigenvectors so that the corresponding orthogonal matrix is a rotation and not a reflection (by making the determinant of the matrix $+1$ instead of -1).

$$\boldsymbol{B} = \left\{ \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \right\} \qquad P = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\cancel{A = P^{-1} D P = P^T D P}$$

$$AP = DP$$

$$\boxed{P^{-1}AP = D}$$

$$\boxed{P^T AP = D}$$

Note

$$P = \boldsymbol{P}_{E \leftarrow B}$$

where as always,

$$E = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}.$$

For $\underline{\boldsymbol{v}} \in \mathbb{R}^2$ write $\underline{\boldsymbol{v}} = \begin{bmatrix} x \\ y \end{bmatrix}$ in standard coordinates and $[\underline{\boldsymbol{v}}]_B = \begin{bmatrix} x' \\ y' \end{bmatrix}$. (The text uses $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ respectively.) So the two coordinate systems are related by

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}.$$

Do algebra!

$$2x^2 + 2y^2 - 5xy = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 2 & \frac{5}{2} \\ \frac{5}{2} & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} x' & y' \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & +\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 2 & \frac{5}{2} \\ \frac{5}{2} & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$= \begin{bmatrix} x' & y' \end{bmatrix} \begin{bmatrix} \frac{9}{2} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} \quad \text{(because } P^T AP = \text{D)}$$

$$= \frac{9}{2}(x')^2 - \frac{1}{2}(y')^2.$$

So the original curve with equation

$$2x^2 + 2y^2 + 5xy = 1$$

in the standard coordinate system has equation

$$\frac{9}{2}(x')^2 - \frac{1}{2}(y')^2 = 1$$

with respect to the rotated coordinate system!

Answer to 1a) This curve is a hyperbola! In the rotated coordinate system its equation is

$$\frac{(x')^2}{\left(\sqrt{\frac{2}{3}}\right)^2} - \frac{(y')^2}{(\sqrt{2})^2} = 1.$$

Answer to 1b) No! $f(x, y) = 2x^2 + 2y^2 + 5xy$ does not have a local min or max at $(0, 0)$. The origin is a saddle point, because in the rotated coordinate system

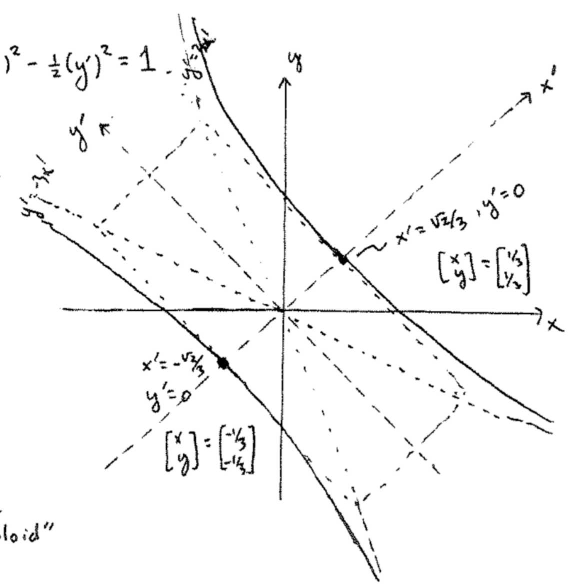
$$f(x', y') = \frac{9}{2}(x')^2 - \frac{1}{2}(y')^2.$$

Old pictures from when I could still sketch well:

$$= \begin{bmatrix} x' & y' \end{bmatrix} \begin{bmatrix} 9/2 & 0 \\ 0 & -1/2 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = \frac{9}{2}(x')^2 - \frac{1}{2}(y')^2 = 1.$$

ans to 1

Curve is a hyperbola! $\frac{(x')^2}{(\sqrt{2/3})^2} - \frac{(y')^2}{(\sqrt{2})^2} = 1$

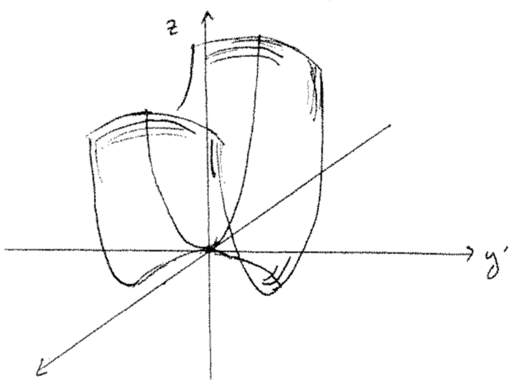


NO! not a local min!

ans to 2 $\quad z = 2x^2 + 2y^2 + 5xy$

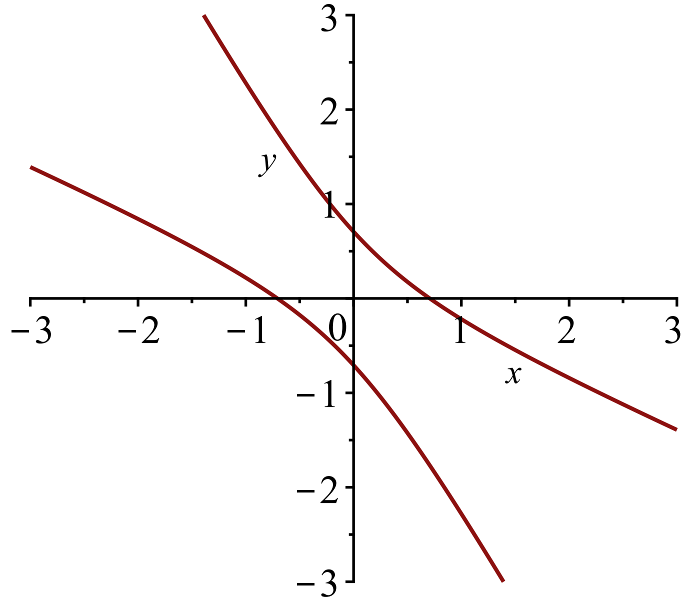
$z = \frac{9}{2}(x')^2 - \frac{1}{2}(y')^2$ $\quad$ Saddle surface!

"parabolic hyperboloid"



Maple verification: To be continued ....

> $with(plots):$
$implicitplot\left(2\cdot x^2 + 2\cdot y^2 + 5\cdot x\cdot y = 1, x = -3..3, y = -3..3, grid = [200, 200]\right);$



> $plot3d\left(2\cdot x^2 + 2\cdot y^2 + 5\cdot x\cdot y, x = -3..3, y = -3..3\right);$