Math 2270-004 Week 14 notes
We will not necessarily finish the material from a given day's notes on that day. We may also add or subtract some material as the week progresses, but these notes represent an in-depth outline of what we plan to cover. These notes cover material in 6.8, 7.1-7.2, with some supplementary material. The Friday notes are not yet included.

Mon Apr 16

- 6.8 Truncated Fourier series as projection of functions via an orthonormal basis of sinusoidal functions; Fourier series in two variables and the idea behind jpg image compression, show and tell.

Announcements:
- does Wolfram alpha work with large matrices? anything else online?
- careful Fourier series
- jpeg compression.

Warm-up Exercise: well, you could review the dot product, inner product flow chart...

Flow chart of dot product development in $\mathbb{R}^n$

$$\vec{x}\cdot\vec{y} := \sum_{i=1}^{n} x_i y_i$$

algebra

a) $\vec{x}\cdot\vec{y} = \vec{y}\cdot\vec{x}$ (symmetry)

b) $\vec{x}\cdot(\vec{y}+\vec{z}) = \vec{x}\cdot\vec{y} + \vec{x}\cdot\vec{z}$ (linear in each factor)

$\quad \vec{x}\cdot(k\vec{y}) = k\,\vec{x}\cdot\vec{y}$

c) $\vec{x}\cdot\vec{x} \geq 0$; $\vec{x}\cdot\vec{x} = 0$ iff $\vec{x} = \vec{0}$ (positive)

An inner product space is a (real scalar) vector space V together with an inner product $\langle\cdot,\cdot\rangle$ which gives a real number for each pair of vectors, s.t. the following axioms hold, $\forall f,g,h \in V$, $k \in \mathbb{R}$:

- a) $\langle f,g\rangle = \langle g,f\rangle$
- b) $\langle f,g+h\rangle = \langle f,g\rangle + \langle f,h\rangle$
- $\quad \langle f,kg\rangle = k\langle f,g\rangle$
- c) $\langle f,f\rangle \geq 0$. $\langle f,f\rangle = 0$ iff $f = 0$

From these algebra axioms the entire concept chart on the left also holds, for finite dimensional subspaces W.

from algebra...

$\|\vec{x}\|^2 = \vec{x}\cdot\vec{x}$

magnitude (norm)

$\vec{x}\cdot\vec{y} = 0$

orthogonal

$\mathrm{proj}_L \vec{x}$

$L = \mathrm{span}\{\vec{v}\}$

$\vec{x}\cdot\vec{y} = \|\vec{x}\|\|\vec{y}\|\cos\theta$ (angles)

distance from $\vec{x}$ to $\vec{y}$ $= \|\vec{y}-\vec{x}\|$

Pythagorean Theorem

orthonormal basis for $W \subset \mathbb{R}^n$, coords for such basis

Gram-Schmidt algorithm

$|\vec{x}\cdot\vec{y}| \leq \|\vec{x}\|\|\vec{y}\|$ Cauchy Schwarz inequality

$\|\vec{x}+\vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$

triangle inequality (for estimates)

projection onto W

$\mathbb{R}^n = W \oplus W^\perp$

- i.e. $\vec{x} = \underbrace{\mathrm{proj}_W \vec{x}}_{\in W} + \underbrace{\vec{z}}_{\in W^\perp}$, uniquely

nearest point to $\vec{b} \in \mathbb{R}^n$ in W is $\mathrm{proj}_W \vec{b}$

Least squares solutions to $A\vec{x} = \vec{b}$

data fitting

linear regression

polynomial fits

power laws.

Example for the inner product on $C[-\pi, \pi]$ given by

$$\langle f, g\rangle := \frac{1}{\pi}\int_{-\pi}^{\pi} \cancel{f(t)\,dt}\; f(t)\,g(t)\,dt$$

The infinite set of functions

$$\left\{\frac{1}{\sqrt{2}}, \cos(t), \sin(t), \cos(2\,t), \sin(2\,t), \ldots, \sin(nt), \cos(nt), \ldots..\right\}$$

$$\left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\,dt = \frac{1}{\pi}\frac{1}{2}\int_{-\pi}^{\pi} 1\,dt = \frac{2\pi}{2\pi} = 1$$

is already orthonormal! Thus begins the subject of *Fourier Series*. (See Wikipedia.)

To show the ortho-normality properties one applies the following trig identities, which follow from the addition angle formulas

$$\cos(m\,t)\cos(n\,t) = \frac{1}{2}\left[\cos((m+n)t) + \cos((m-n)\,t)\right]$$

$$\cos^2(n\,t) = \frac{1}{2}\left[\cos(2\,n\,t) + 1\right]$$

$$\sin(m\,t)\sin(n\,t) = \frac{1}{2}\left[-\cos((m+n)\,t) + \cos((m-n)t)\right]$$

$$\sin^2(n\,t) = \frac{1}{2}\left[-\cos(2\,n\,t) + 1\right]$$

$$\cos(m\,t)\sin(n\,t) = \frac{1}{2}\left[\sin((m+n)t) + \sin((-m+n)t)\right]$$

Exercise verify how ortho-normality follows from these identities.

$$\langle \cos mt, \cos nt \rangle = 0 \qquad m \neq n.$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \cos(mt)\cos(nt)\,dt = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1}{2}\left(\cos(m+n)t + \cos(m-n)t\right)dt$$

$$= \frac{1}{\pi}\left[\frac{1}{2}\left(\frac{\sin(m+n)t}{m+n} + \frac{\sin(m-n)t}{m-n}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}(0-0) = 0$$

$$\|\cos nt\|^2 = \langle \cos nt, \cos nt\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{\frac{1}{2}\left[\cos 2nt + 1\right]}_{\cos^2 nt} dt = \frac{1}{\pi}\frac{1}{2}\left(\frac{\sin 2nt}{2n} + t\right)\Bigg]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\frac{1}{2}\left(0 + \pi - (0 - \pi)\right)$$

$$= \frac{2\pi}{2\pi} = 1$$

Let $V_n := span\left\{\frac{1}{\sqrt{2}}, \cos(t), \sin(t), \cos(2\,t), \sin(2\,t), \ldots.\ \cos(n\,t), \sin(n\,t)\right\}$ be the $2\,n+1$ dimensional subspace spanned by the first $2\,n+1$ of these functions. A deep theorem says that if $f \in C(-\pi, \pi)$ (actually, f only needs to be piecewise continous), then

$$\lim_{n\to\infty} \left\| f - proj_{V_n} f \right\| = 0.$$

Because we have an orthonormal basis for V_n the projection formula is easy to write down:

$proj_W \vec{x} = (\vec{x}\cdot\vec{u}_1)\vec{u}_1 + (\vec{x}\cdot\vec{u}_2)\vec{u}_2 + \cdots + (\vec{x}\cdot u_N)\vec{u}_N$

$$proj_{V_n} f = \left\langle f, \frac{1}{\sqrt{2}}\right\rangle \frac{1}{\sqrt{2}} + \underbrace{\langle f, \cos(t)\rangle}_{a_1}\cos(t) + \underbrace{\langle f, \sin(t)\rangle}_{b_1}\sin(t) + \ldots. + \langle f, \cos(n\,t)\rangle\cos(n\,t) + \langle f, \sin(n\,t)\rangle\sin(n\,t)$$

$$\langle f, g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)g(t)\,dt$$

.

We write

$$a_0 = \langle f, 1\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,\mathrm{d}t$$

$$a_k = \langle f, \cos(k\,t)\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(k\,t)\,\mathrm{d}t$$

$$b_k = \langle f, \sin(k\,t)\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(k\,t)\,\mathrm{d}t.$$

Then

$$proj_{V_n} f = \frac{a_0}{2} + \sum_{k=1}^{n} a_k\cos(k\,t) + \sum_{k=1}^{n} b_k\sin(k\,t).$$

The infinite series converges to $f(t)$ pointwise at places where f is differentiable, and to the average of right and left hand limits at jump discontinuities, so we also often consider the infinite Fourier series

$$f(t) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k\cos(k\,t) + \sum_{k=1}^{\infty} b_k\sin(k\,t).$$

$$f(t) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kt) + \sum_{k=1}^{\infty} b_k \sin(kt).$$

$$a_0 = \langle f, 1 \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,\mathrm{d}t \qquad a_k = \langle f, \cos(kt) \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(kt)\,\mathrm{d}t$$

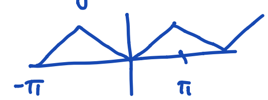
$$b_k = \langle f, \sin(kt) \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(kt)\,\mathrm{d}t.$$

Exercise: Define $f(t) = t$, on the interval $-\pi < t < \pi$. Show

$$t \sim 2 \sum_{k=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nt)$$

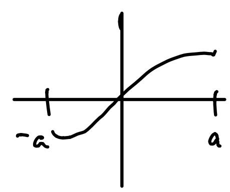
n

On HW $g(t) = |t|$ $-\pi < t < \pi$



$$a_0 = \langle f, 1 \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{t \cdot 1}\, dt = 0$$

$\boxed{g(t) = t \text{ is } \underline{\text{odd}} \quad g(-t) = -g(t)}$



$$\int_{-a}^{a} g(t)\,dt = 0.$$

finish on Monday!

$$a_n = \langle f, \cos nt \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\, dt$$

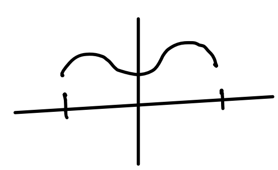
$$= \frac{1}{\pi}\int_{-\pi}^{\pi} t \cos nt\, dt = 0$$

integral of odd fcn over interval $[-a,a]$ is zero.

odd fcn

even fcn

$h(t)$ even means $h(-t) = h(t)$



$g(t)$ odd

$h(t)$ even

$g(t)h(t)$ is odd

$g(-t)h(-t) = -g(t)h(t)$

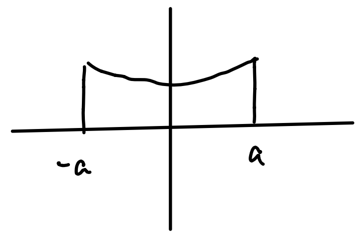
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{\underbrace{t}_{\text{odd}}\ \underbrace{\sin(nt)}_{\text{odd}}}_{\text{even}}\, dt$$

if $g(t), h(t)$ are odd

then $g(t)\,h(t)$ is even

$$g(-t)\,h(-t) = (-g(t))(-h(t))$$
$$= g(t)\,h(t)$$

if $f(t)$ even $\displaystyle\int_{-a}^{a} f(t)\,dt = 2\int_{0}^{a} f(t)\,dt$



$$b_n = \frac{1}{\pi}\,2\int_0^{\pi} \underbrace{t}_{u}\ \underbrace{\sin nt\, dt}_{dv}$$

$du = dt \qquad v = -\dfrac{\cos nt}{n}$

$$= \frac{2}{\pi}\left[uv - \int v\,du\right]$$

$$= \frac{2}{\pi}\left[\ t\left(-\frac{\cos nt}{n}\right)\Big]_0^{\pi} - \int_0^{\pi}\left(-\frac{\cos nt}{n}\right)dt\right]$$

$$\cancel{-\frac{\sin nt}{n^2}\Big]_0^{\pi}} \rightarrow 0$$

$$= \frac{2}{\cancel{\pi}}\left[\cancel{\pi}\left(-\frac{\cos n\pi}{n}\right) - 0\right]$$

$$= \frac{2}{n}(-\underbrace{\cos n\pi}_{(-1)^n})$$

$$\boxed{b_n = \frac{2}{n}(-1)^{n+1}}$$

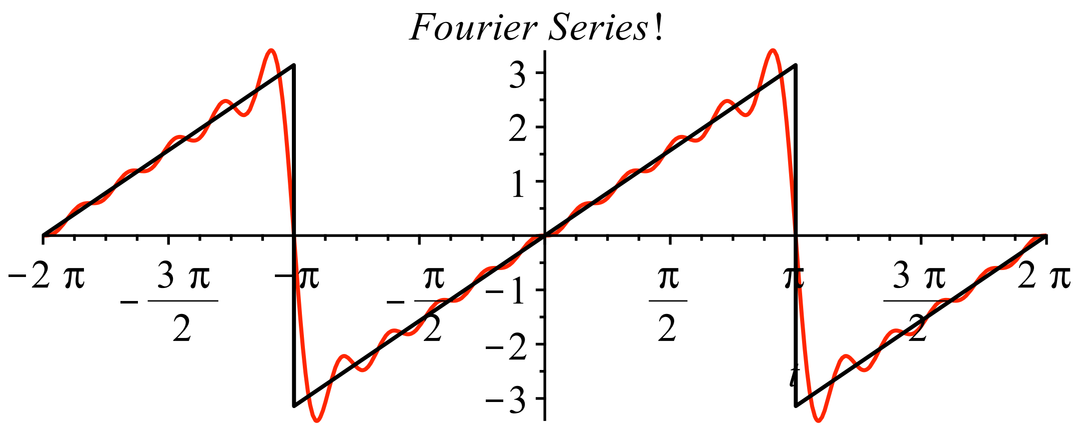
$$\boxed{\text{"}t\text{"} = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\sin nt}$$

$$\| t - \operatorname{proj}_{J_n} t \| \to 0$$

$$\mathrm{proj}_{V_{10}}\, f(t):$$

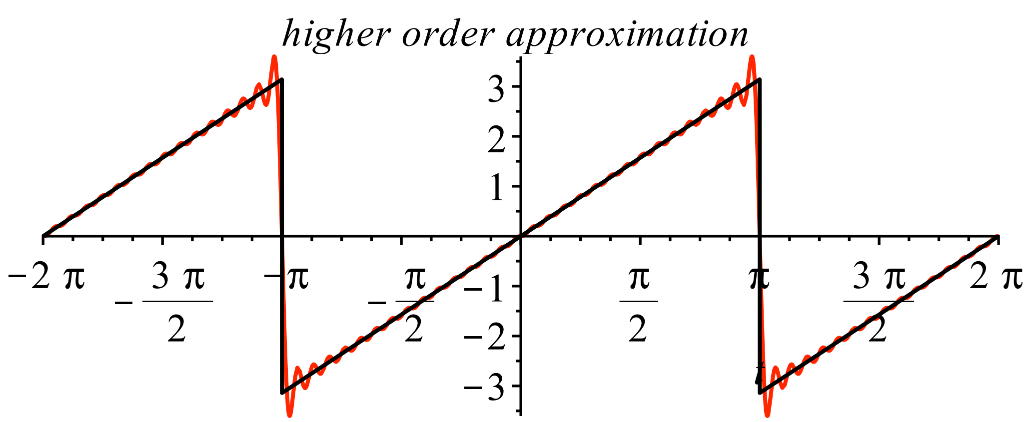
```
> with(plots):
  plot1 := plot(t + 2·π − 2·π·Heaviside(t + π) − 2·π·Heaviside(t − π), t=−2·π..2·π, color
      = black):
  plot2 := plot(2·sum((−1)^(n+1)·sin(n·t)/n, n=1..10), t=−2·π..2·π, color = red):
  display({plot1, plot2}, title = 'Fourier Series!');
```

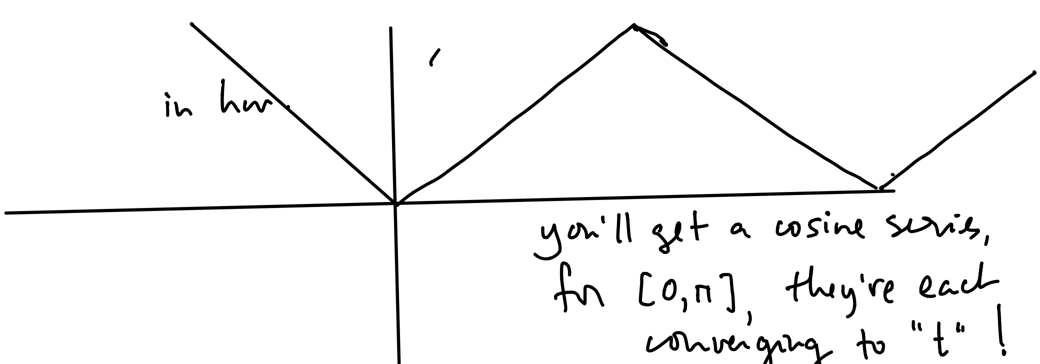
$$plot2 := plot\left(2\cdot\sum_{n=1}^{10}(-1)^{n+1}\cdot\frac{\sin(n\cdot t)}{n},\ t=-2\cdot\pi..2\cdot\pi,\ color=red\right):$$




```
> plot3 := plot(2·sum((−1)^(n+1)·sin(n·t)/n, n=1..30), t=−2·π..2·π, color = red):
  display({plot1, plot3}, title = 'higher order approximation');
```

$$plot3 := plot\left(2\cdot\sum_{n=1}^{30}(-1)^{n+1}\cdot\frac{\sin(n\cdot t)}{n},\ t=-2\cdot\pi..2\cdot\pi,\ color=red\right):$$





you'll get a cosine series,
for $[0,\pi]$, they're each
converging to "t"!

As part of the deep theorem about Fourier series, as long as f is piecewise continuous,

$$\left\| proj_{V_N} f - f \right\| \to 0 \quad \text{and} \left\| proj_{V_N} f \right\| \to \| f \| .$$

Recall, the norm that we get from the Fourier series inner product is

$$\| g \|^2 = \frac{1}{\pi} \int_{-\pi}^{\pi} g(t)^2 \, dt.$$

Now,

$$proj_{V_n} f = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos(k\,t) + \sum_{k=1}^{n} b_k \sin(k\,t)$$

So

$$\left\| proj_{V_n} f \right\|^2 = \left\| \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos(k\,t) + \sum_{k=1}^{n} b_k \sin(k\,t) \right\|^2$$

$\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ ortho normal

$\to (c_1\vec{u}_1 + c_2\vec{u}_2 + c_3\vec{u}_3) \cdot (c_1\vec{u}_1 + c_2\vec{u}_2 + c_3\vec{u}_3)$
$= c_1^2 + c_2^2 + c_3^2$

$$= \left\langle \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos(k\,t) + \sum_{k=1}^{n} b_k \sin(k\,t), \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos(k\,t) + \sum_{k=1}^{n} b_k \sin(k\,t) \right\rangle$$

$$= \left\langle \frac{a_0}{2}, \frac{a_0}{2} \right\rangle + \sum_{k=1}^{n} a_k^2 + \sum_{k=1}^{n} b_k^2 = \frac{a_0^2}{2} + \sum_{k=1}^{n} a_k^2 + \sum_{k=1}^{n} b_k^2,$$

because the cross terms in the expanded inner product cancel out - since the basis vectors we've chosen for V_n are orthonormal:

$$V_n := span\left\{ \frac{1}{\sqrt{2}}, \cos(t), \sin(t), \cos(2\,t), \sin(2\,t), \ldots\ \cos(n\,t), \sin(n\,t) \right\}$$

$\int t^2 = \frac{t^3}{3}$

As an application, for our function $f(t) = t$,

$$\| f \|^2 = \frac{1}{\pi} \int_{-\pi}^{\pi} t^2 \, dt = \frac{2}{\pi} \int_{0}^{\pi} t^2 \, dt = \frac{2}{3} \pi^2. \qquad \frac{2}{\pi} \left. \frac{t^3}{3} \right]_0^{\pi} = \frac{2}{3} \pi^2$$

Since

$$t \sim 2 \sum_{k=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(n\, t)$$

$b_n = \frac{2(-1)^{n+1}}{n} \qquad b_n^2 = \frac{4}{n^2}$

It must be that

$$\frac{2}{3} \pi^2 = 4 \sum_{k=1}^{\infty} \frac{1}{n^2}$$

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \sum_{k=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

This magic formula is true (which is sort of amazing), although you may not have seen it before:

> $\sum_{k=1}^{\infty} \frac{1}{k^2}$;

$$\frac{1}{6} \pi^2 \qquad \textbf{(1)}$$

·jpg image compression show & tell. More info at Wikipedia

Prof. Nesse

Math 3150 2D Fourier Series Field Trip Project

Due date: Monday, Nov. 28 after Thanksgiving holiday.

Two-dimensional Fourier series can be used to perform image processing and data compression, and is a salient example of how an set of orthogonal basis functions can be used to approximate functions. Suppose $f(x, y)$ is defined on the region $(x, y) \in [0, L] \times [0, H]$ and represents a grey scale image. For each point (x, y) the greyscale value ranges from zero (black) to unity (white) $f \in [0, 1]$. The orthogonal basis set we use is a 2D sine series:

$$\phi_{n,m}(x, y) = \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{H}y\right),$$

← .jpg actually uses cosine series

where n and m both range from $1, 2, 3, \ldots$. The values $\frac{n\pi}{L}$ and $\frac{m\pi}{H}$ represent the horizontal and vertical spatial frequencies. The approximate image is the double sum orthogonal projection:

$$\hat{f}_{N,M}(x, y) = \sum_{n=1}^{N}\sum_{m=1}^{M} B_{n,m}\phi_{n,m}(x, y),$$

where N and M represent the sum truncation and the values $\frac{N\pi}{L}$ and $\frac{M\pi}{H}$ represent the maximum horizontal and vertical spatial frequencies available to represent the image—any image feature that has higher spatial frequency, such as sharp areas of contrast, fine texture, etc, cannot be represented. The Fourier coefficients are

$$B_{n,m} = \frac{\langle f, \phi_{n,m}\rangle}{\langle \phi_{n,m}, \phi_{n,m}\rangle}, \quad \langle g, h\rangle = \int_0^L \int_0^H g(x, y)h(x, y)dydx.$$

The goal of the project is to assess the qualitative nature of Fourier image processing in two experiments.

Experiment 1:

The first experiment you will show a subject (a friend that has not seen the full image) a Fourier-decomposed image of a car/truck that you take with your cell phone camera. I advise to use a "square" Instagram-ready image, and compress it to 256X256 pixels, which is most easily accomplished by re-sending the picture to yourself by email and compressing it to the "small" size for sending. The automobile image should be centered in the fame, and fill approximately 1/2-3/4 the width of the frame. It should be a random car parked on a street, or something, that's from a common brand and model that's recognizable to most people, or at least your friend. Your friend should not know anything about the picture at all and don't tell them anything. Show your friend successively higher truncated Fourier compressed images of the car until your friend correctly guesses (1) that its a automobile of some type, and then (2) guesses the make and/or model type. Start by showing your friend the Fourier compressed image at $N = M = 10$, then ascend $N = 20, 30, 40, 50, \ldots$ until he/she gets both (1) and then (2). At each stage, record you friend's response and report your results, including the images and the compressed images.

Experiment 2:

Take two pictures, both 256X256 as described above. One of the images should be a "natural scene," which should be interpreted broadly as naturescapes, or varied urban cityscapes—the point is that it should contain a mix lots of things in the image, both foreground and background, objects with lots of different sizes in the frame—and the other should be a somewhat boring picture of a single human-made material—e.g., a wall of bricks, patterned fabric, things of a regular or repeated nature to it; be sure that you fit several repetitions of the pattern in your picture.. Get inventive with what you choose. We will compare the two pictures' Fourier coefficients $B_{n,m}$. Natural images have been commonly reported to have squared Fourier coefficients that decay with a power law:

$$B_{n,m}^2 \sim \frac{1}{n^\gamma} \quad \text{or} \quad \sim \frac{1}{m^\gamma},$$

where γ is typically in the range between 1.7 and 2.3. That is, γ is usually around 2. Why do we examine squared Fourier coefficients? Its because squared values are associated with the energy in the image through Parseval's identity:

$$Energy = \int_0^L \int_0^H |f(x,y)|^2 dydx = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} B_{n,m}^2.$$

In the accompanying code, the coefficients $B_{n,m}$ are computed and represented as a matrix, and rendered of the squared values of the $B_{n,m}$. The energy spectrum is a log-log plot of the average of vertical and horizontal average energies: $\frac{1}{2} avg_m(B_{n,m}^2) + \frac{1}{2} avg_m(B_{m,n}^2) = b_n^2$—this gives an estimate of the spectral energy at each spatial frequency $n\pi$ per unit image length L. If $b_n^2 \sim \frac{1}{n^\gamma}$, then taking the log of both sides we get:

$$\ln(b_n^2) \sim \ln\left(\frac{1}{n^\gamma}\right) = -\gamma \ln(n).$$

That is, the log squared coefficient averages will be linearly related to the log of n with slope $-\gamma$. The code performs a linear curve fitting on the log-log data and finds the best-fit γ-value as our estimate. For the two images you choose, record the γ-estimates and report them in your results along with your images. Use a large truncation $N = 100$ value—it may take a while. Report the gamma-values you find, and the standard deviations of the linear fit.

How to use the code: Put the images you want to analyze in a file folder with the .m code given with this experiment. Type in the code the file name of the image you want to examine and edit the code to set an N value for your truncation. There is a variable called "experiment", which you set to 1 or 2, respectively. The code will output figures. Figure 1 will give you the Nth Fourier truncated image in greyscale. Figure 2 will output the results for experiment 2.