Fri Apr 13

- 6.7-6.8 Introduction to inner product spaces.

Announcements: .jpg show & tell on Monday I hope.

Warm-up Exercise: look over HW assignment

Flow chart of dot product development in $\mathbb{R}^n$

$$\vec{x}\cdot\vec{y} := \sum_{i=1}^{n} x_i y_i$$

↓ algebra

a) $\vec{x}\cdot\vec{y} = \vec{y}\cdot\vec{x}$ (symmetry)

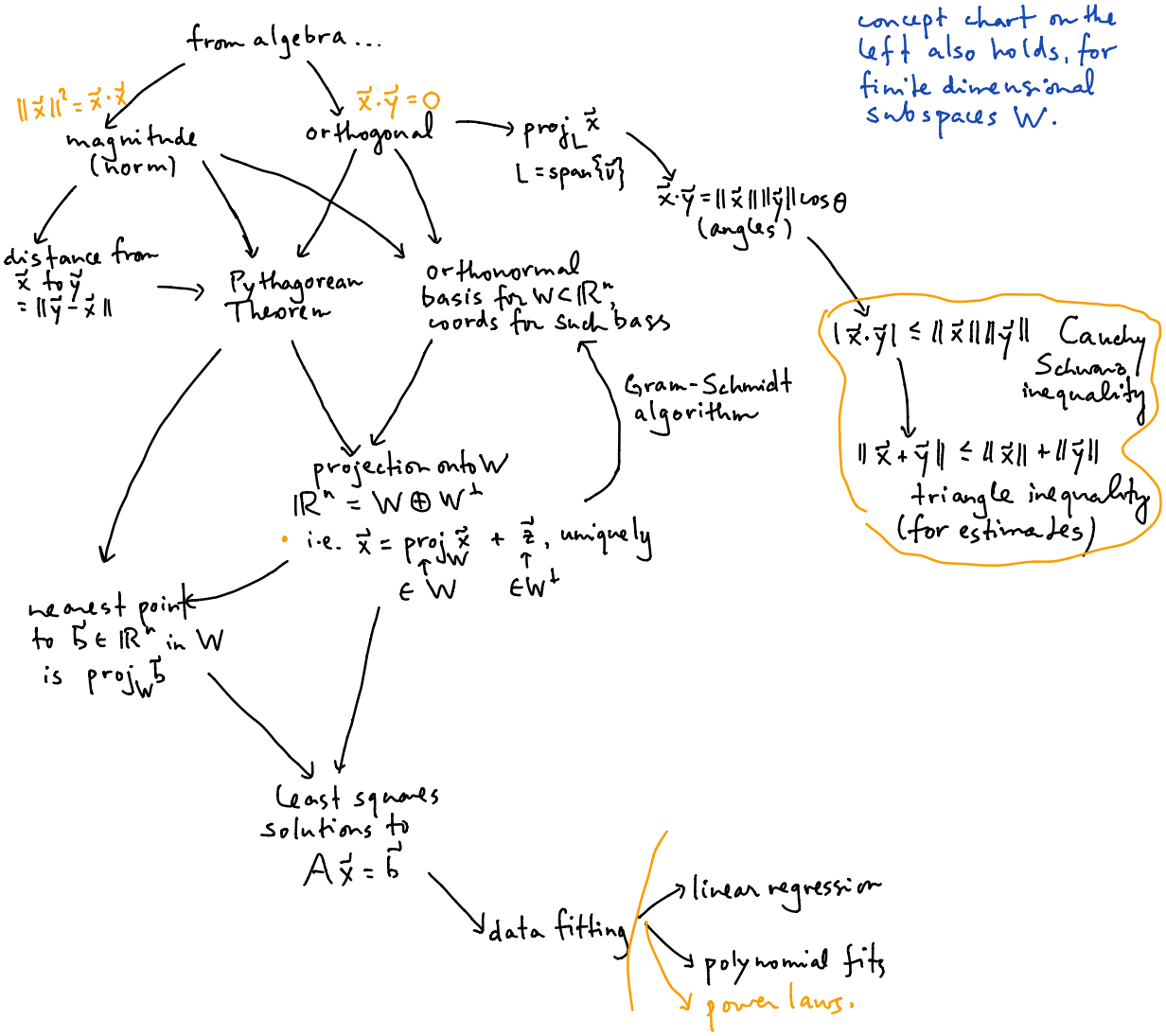
b) $\vec{x}\cdot(\vec{y}+\vec{z}) = \vec{x}\cdot\vec{y} + \vec{x}\cdot\vec{z}$, $\vec{x}\cdot(k\vec{y}) = k\,\vec{x}\cdot\vec{y}$ (linear in each factor)

c) $\vec{x}\cdot\vec{x} \geq 0$; $\vec{x}\cdot\vec{x} = 0$ iff $\vec{x} = \vec{0}$ (positive)

An inner product space is a (real scalar) vector space V together with an inner product $\langle \cdot, \cdot \rangle$ which gives a real number for each pair of vectors, s.t. the following axioms hold, $\forall f, g, h \in V$, $k \in \mathbb{R}$:

- a) $\langle f, g \rangle = \langle g, f \rangle$
- b) $\langle f, g+h \rangle = \langle f, g \rangle + \langle f, h \rangle$
  $\langle f, kg \rangle = k \langle f, g \rangle$
- c) $\langle f, f \rangle \geq 0$. $\langle f, f \rangle = 0$ iff $f = 0$

From these algebra axioms the entire concept chart on the left also holds, for finite dimensional subspaces W.



Examples of function space inner products:

$$V = \{f : [a, b] \to \mathbb{R} \; s.t. \; f \; is \; continuous\} := C([a, b]).$$

$$\langle f, g \rangle := \int_a^b f(t)g(t)\,dt \quad \text{(or some fixed positive multiple of this integral).}$$

Exercise 1) Check the algebra requirements a), b), c) for an inner product.

$\vec{x}\cdot\vec{y} = \vec{y}\cdot\vec{x}$ a) $\langle f, g \rangle = \langle g, f \rangle$ $\quad \int_a^b f(t)g(t)\,dt = \int_a^b g(t)f(t)\,dt$

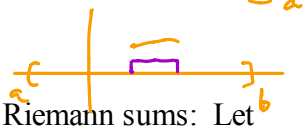
$\vec{x}\cdot(\vec{y}+\vec{z}) = \vec{x}\cdot\vec{y} + \vec{x}\cdot\vec{z}$ b) $\langle f, g+h \rangle = \langle f, g \rangle + \langle f, h \rangle$ $\quad \int_a^b f(t)(g(t)+h(t))\,dt = \int_a^b f(t)g(t)\,dt + \int_a^b f(t)h(t)\,dt$ ✓

c) $\langle f, f \rangle \geq 0$, $\langle f, f \rangle = 0$ if only if $f = 0$

$\int_a^b f(t)^2\,dt \geq 0.$

if f is continuous & $\int_a^b f(t)^2\,dt = 0$

$\Rightarrow f = 0$:



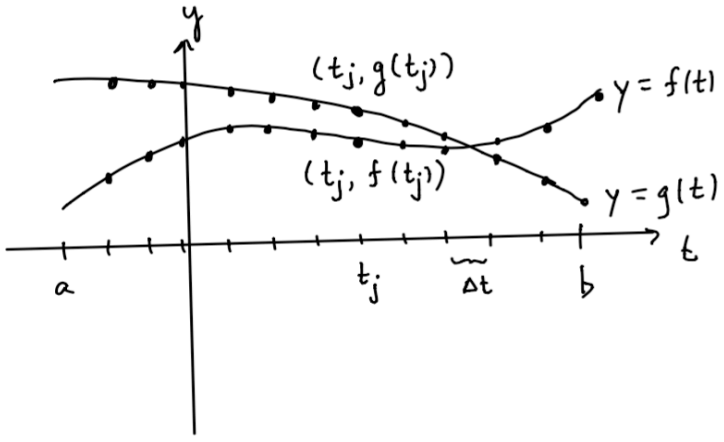
This inner product $\langle f, g \rangle$ is not so different from the $\mathbb{R}^n$ dot product if you think of Riemann sums: Let

$$\Delta t = \frac{b-a}{n}; \qquad t_j = a + j\,\Delta t, j = 1, 2, .. n.$$

Then

$$\langle f, g \rangle = \int_a^b f(t)g(t)\,dt = \lim_{n\to\infty} \sum_{j=1}^{n} f(t_j)\, g(t_j)\Delta t \quad = \frac{b-a}{n}$$

$$= \lim_{n\to\infty} \left(\begin{bmatrix} f(t_1) \\ f(t_2) \\ \vdots \\ f(t_n) \end{bmatrix} \cdot \begin{bmatrix} g(t_1) \\ g(t_2) \\ \vdots \\ g(t_n) \end{bmatrix} \Delta t \right).$$



Prime examples:

Example For the inner product on $C[-1, 1]$ given by

$$\langle f, g \rangle := \int_{-1}^{1} f(t)g(t)\, dt$$

If one applies Gram-Schmidt to the the set $\{1, t, t^2, t^3, \ldots.\}$ one creates the (normalized) *Legendre polynomials* which have an interesting entry at Wikipedia. Projecting a continuous function f onto

$$W_n = span\{1, t, t^2, \ldots t^n\}$$

will create polynomical approximations, that improve in the sense that

$$\lim_{n \to \infty} \left\| f - proj_{W_n} f \right\|^2 = 0.$$

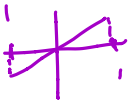
$w_1\ w_2\ w_3$

$\vec{u}_1 = \frac{\vec{w}_1}{\|\vec{w}_1\|}$ $\qquad \|\vec{w}_1\| = \langle \vec{w}_1, \vec{w}_1 \rangle^{1/2} = \left(\int_{-1}^{1} (1)(1)\, dt \right)^{1/2} = \sqrt{2}$

$w_1 =$ "1"

$\vec{u}_1 =$ is the fcn $u_1(t) = \frac{1}{\sqrt{2}}$

$\vec{w}_2 \perp \vec{w}_1$ or $\vec{u}_1$ $\qquad \vec{z}_2 = \vec{w}_2 - (\vec{w}_2 \cdot \vec{u}_1)\vec{u}_1$



$\vec{w}_2 \cdot \vec{u}_1 = \int_{-1}^{1} t \frac{1}{\sqrt{2}}\, dt = \frac{t^2}{2\sqrt{2}} \Big]_{-1}^{1} = 0$

$\vec{u}_2 = \frac{\text{"}t\text{"}}{\|\text{"}t\text{"}\|}$ ...

Example for the inner product on $C[-\pi, \pi]$ given by

$$\langle f, g \rangle := \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\, g(t)\, dt$$

The infinite set of functions

$$\left\{ \frac{1}{\sqrt{2}}, \cos(t), \sin(t), \cos(2t), \sin(2t), \ldots, \sin(nt), \cos(nt), \ldots \right\}$$

is already orthonormal! Thus begins the subject of *Fourier Series*. (See Wikipedia.)

To show the ortho-normality properties one applies the following trig identities, which follow from the addition angle formulas

$$\cos(mt)\cos(nt) = \frac{1}{2}[\cos((m+n)t) + \cos((m-n)t)]$$

$$\cos^2(nt) = \frac{1}{2}[\cos(2nt) + 1]$$

$$\sin(mt)\sin(nt) = \frac{1}{2}[-\cos((m+n)t) + \cos((m-n)t)]$$

$$\sin^2(nt) = \frac{1}{2}[-\cos(2nt) + 1]$$

$$\cos(mt)\sin(nt) = \frac{1}{2}[\sin((m+n)t) + \sin((-m+n)t)]$$

Exercise verify how ortho-normality follows from these identities.

$\langle \cos mt, \cos nt \rangle = 0 \quad m \neq n.$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \cos(mt)\cos(nt)\, dt = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1}{2}\left(\cos(m+n)t + \cos(m-n)t\right) dt$$

$$= \frac{1}{\pi}\left[\frac{1}{2}\left(\frac{\sin(m+n)t}{m+n} + \frac{\sin(m-n)t}{m-n}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}(0-0) = 0$$

$$\|\cos nt\|^2 = \langle \cos nt, \cos nt \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{\frac{1}{2}[\cos 2nt + 1]}_{\cos^2 nt}\, dt = \frac{1}{\pi}\frac{1}{2}\left(\frac{\sin 2nt}{2n} + t\right)\Big]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\frac{1}{2}(0 + \pi - (0 - \pi))$$

$$= \frac{2\pi}{2\pi} = 1$$

Let $V_n := span\left\{\frac{1}{\sqrt{2}}, \cos(t), \sin(t), \cos(2t), \sin(2t), \ldots \cos(nt), \sin(nt)\right\}$ be the $2n+1$ dimensional subspace spanned by the first $2n+1$ of these functions. A deep theorem says that if $f \in C(-\pi, \pi)$ (actually, f only needs to be piecewise continous), then

$$\lim_{n\to\infty} \left\| f - proj_{V_n} f \right\| = 0.$$

Because we have an orthonormal basis for V_n the projection formula is easy to write down:

$$proj_{V_n} f = \left\langle f, \frac{1}{\sqrt{2}} \right\rangle \frac{1}{\sqrt{2}} + \langle f, \cos(t)\rangle \cos(t) + \langle f, \sin(t)\rangle \sin(t) + \ldots + \langle f, \cos(nt)\rangle \cos(nt) + \langle f, \sin(nt)\rangle \sin(nt)$$

We write

$$a_0 = \langle f, 1\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\, dt$$

$$a_k = \langle f, \cos(kt)\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(kt)\, dt$$

$$b_k = \langle f, \sin(kt)\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(kt)\, dt.$$

Then

$$proj_{V_n} f = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos(kt) + \sum_{k=1}^{n} b_k \sin(kt).$$

The infinite series converges to $f(t)$ pointwise at places where f is differentiable, and to the average of right and left hand limits at jump discontinuities, so we also often consider the infinite Fourier series

$$f(t) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kt) + \sum_{k=1}^{\infty} b_k \sin(kt).$$

$$f(t) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(k\,t) + \sum_{k=1}^{\infty} b_k \sin(k\,t).$$

$$a_0 = \langle f, 1\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\ \mathrm{d}t \qquad a_k = \langle f, \cos(k\,t)\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(k\,t)\ \mathrm{d}t$$

$$b_k = \langle f, \sin(k\,t)\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\ \sin(k\,t)\ \mathrm{d}t.$$

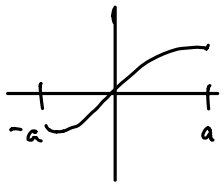
Exercise: Define $f(t) = t$, on the interval $-\pi < t < \pi$. Show

$$t \sim 2\sum_{k=1}^{\infty} \frac{(-1)^{n+1}}{n}\sin(n\,t)$$

$a_0 = \langle f, 1\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} t\cdot 1\,dt = 0$

$g(t) = t$ is odd

$g(-t) = -g(t)$



$\int_{-a}^{a} g(t)\,dt = 0.$

finish on Monday!

$$proj_{V_{10}} f(t):$$

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> with(plots) :
  plot1 := plot(t + 2·π − 2·π·Heaviside(t + π) − 2·π·Heaviside(t − π), t = -2·π..2·π, color
      = black) :
  plot2 := plot(2·Σ_{n=1}^{10} (-1)^{n+1}·sin(n·t)/n, t = -2·π..2·π, color = red) :
  display({plot1, plot2}, title = 'Fourier Series!');
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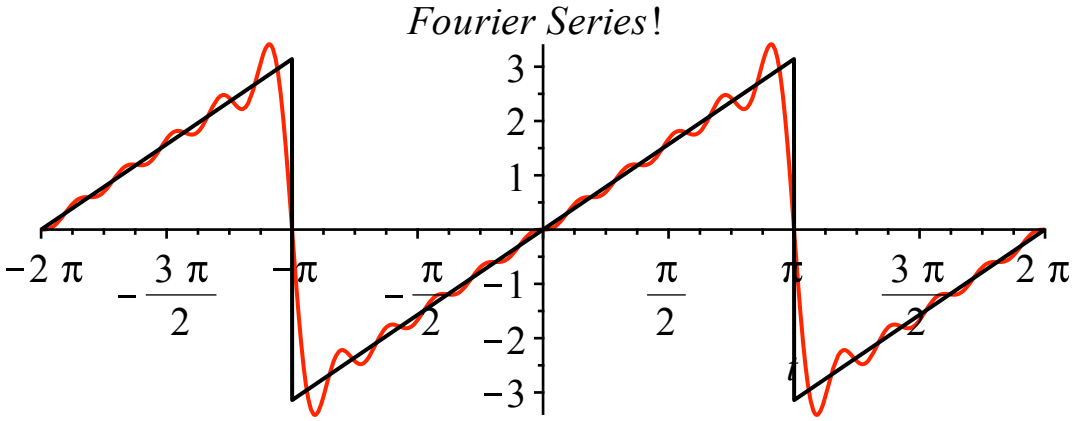


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