Math 2270-002 Week 7 notes

We will not necessarily finish the material from a given day's notes on that day. We may also add or subtract some material as the week progresses, but these notes represent an in-depth outline of what we plan to cover. These notes cover material in 4.1-.4.3.

Mon Oct 1

- 4.1 Vector spaces and subspaces.

Announcements: • Some §4.1 HW: (1) (3) (5) (7) (13) (19) (31) more to come!

• Chapter 4 is "abstract" material about "vector spaces" ↑ collections of objects you can add & scalar multiply.

• next HW is due Wed after break (I will fix it on CANVAS)

• there is a quiz this Wednesday, though.

until 1:00 → try this one!

Warm-up Exercise:

Let $f(t) = \sin t$

$g(t) = t$

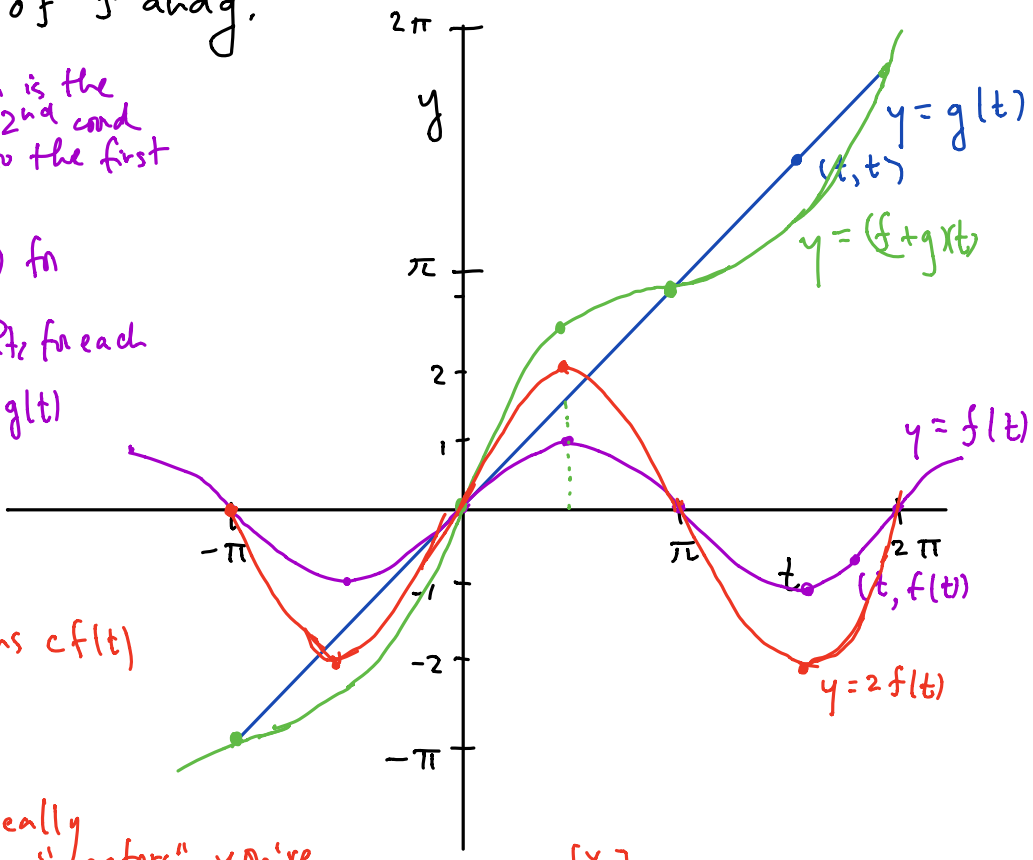
graph $f+g$, $2f$ using the graphs of $f$ and $g$.

key point: graph of fcn is the set of pts where the 2nd coord is the fcn applied to the first coord.

vertical displacement (at $t$) for graph of $f+g$ is the sum of the vertical displacements for each

$(f+g)(t)$ means $f(t)+g(t)$

graph of $cf$ has vertical displacements $c$ times those of $f$, because $(cf)(t)$ means $cf(t)$




functions really are like the "vectors" you're used to — instead of $n$ components you have oo'ly many, one for each $t$ $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$

We dealt with vector equations - linear combination equations - a lot, in Chapters 1-2. There are other objects, for example functions, that one can add and scalar mutiply. The useful framework that includes these other examples is the abstract concept of a *vector space*. There is a body of results that only depends on the axioms below, and not on the particular vector space in which one is working. So rather than redo those results in different-seeming contexts, we understand them once abstractly, in order to apply them in many places. And, we can visualize the abstractions be relying on our solid foundation in $\mathbb{R}^n$.

Definition A *vector space* is a nonempty set $V$ of objects, called *vectors*, on which are defined two operations, called *addition* and *scalar multiplication,* so that the ten axioms listed below hold. These axioms must hold for all vectors $\underline{\boldsymbol{u}}$, $\underline{\boldsymbol{v}}$, $\underline{\boldsymbol{w}}$ in $V$, and for all scalars $c$, $d \in \mathbb{R}$.

1. The sum of $\underline{\boldsymbol{u}}$ and $\underline{\boldsymbol{v}}$, denoted by $\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}}$, is (also) in $V$ (closure under addition.)

2. $\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}} = \underline{\boldsymbol{v}} + \underline{\boldsymbol{u}}$ (commutative property of addition)

3. $(\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}}) + \underline{\boldsymbol{w}} = \underline{\boldsymbol{u}} + (\underline{\boldsymbol{v}} + \underline{\boldsymbol{w}})$ (associative property of addition)

4. There is a **zero** vector $\underline{\mathbf{0}}$ in $V$ so that $\underline{\boldsymbol{u}} + \underline{\mathbf{0}} = \underline{\boldsymbol{u}}$. (additive identity)

5. For each $\underline{\boldsymbol{u}} \in V$ there is a vector $-\underline{\boldsymbol{u}} \in V$ so that $\underline{\boldsymbol{u}} + (-\underline{\boldsymbol{u}}) = \underline{\mathbf{0}}$. (additive inverses)

6. The scalar multiple of $\underline{\boldsymbol{u}}$ by $c$, denoted by $c\,\underline{\boldsymbol{u}}$ is (also) in $V$. (closure under scalar multiplication)

7. $c(\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}}) = c\,\underline{\boldsymbol{u}} + c\,\underline{\boldsymbol{v}}$ (scalar multiplication distributes over vector addition)

8. $(c + d)\,\underline{\boldsymbol{u}} = c\,\underline{\boldsymbol{u}} + d\,\underline{\boldsymbol{u}}$. (scalar addition distributes over scalar multiplication of vectors)

9. $c\,(d\,\underline{\boldsymbol{u}}) = (c\,d)\,\underline{\boldsymbol{u}}$ (associative property of scalar multiplication)

10. $1\,\underline{\boldsymbol{u}} = \underline{\boldsymbol{u}}$ (multiplicative identity)

The following three algebra rules follow from the first 10, and are also useful:

11) $0\,\underline{\boldsymbol{u}} = \underline{\mathbf{0}}$.

12) $c\,\underline{\mathbf{0}} = \underline{\mathbf{0}}$ .

13) $-\underline{\boldsymbol{u}} = (-1)\,\underline{\boldsymbol{u}}$.

Example 1 $\mathbb{R}^n$, $n \geq 1$ a positive integer, with vector addition and scalar multiplication defined component-wise, and with the zero vector being the vector which has every entry equal to zero. Then the properties above just reduce to properties of real number algebra.

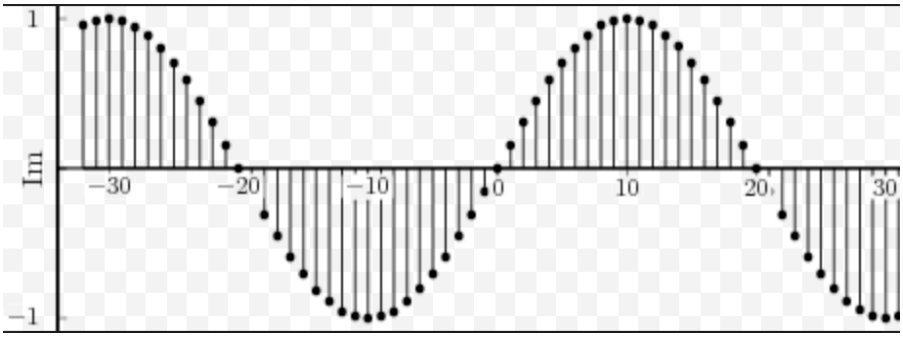
Example 2 The space of $m \times n$ matrices (with $m$, $n$ fixed), with matrix addition and scalar multiplication defined component-wise, and with the zero vector being the matrix which has every entry equal to zero. (If you're only adding and scalar multiplying matrices you can just think of them as rectangularly-shaped vectors.)

Example 3 The set $\mathsf{S}$ of doubly-infinite sequences of numbers. (Think of this as discrete time *signals*.) So an element of $\mathsf{S}$ can be written as

$$\{y_k\} = ( \ldots y_{-2}, y_{-1}, y_0, y_1, y_2, y_3, \ldots )$$

If $\{z_k\}$ is another element of , then the sum $\{y_k\} + \{z_k\}$ is the sequence $\{y_k + z_k\}$, and the scalar multiple $c\{y_k\}$ is the sequence $\{c\, y_k\}$. Checking the vector space axioms is the same as it would be $\mathbb{R}^n$, since addition and scalar multiplication are done entry by entry. (There's just infinitely many entries.)

One can visualize such a discrete time signal in terms of its graph over over its integer "domain". How could you visualize "vector" addition and scalar multiplication in this case?



See warm-up discussion

Example 4 Let $V$ be the set of all real-value functions defined on a domain D on the real line. (In practice D is usually an interval or the entire real line.) If $f$ and $g$ are in $V$, then we define their sum and scalar multiples "component-wise" too, where each $t \in D$ gives a component:

$$(f+g)(t) := f(t) + g(t)$$

$$(cf)(t) := cf(t).$$

So for example, the domain is the entire real line and if $f$ is defined by the rule $f(t) = 1 + 2\sin(3t)$ and $g$ is defined by the rule $g(t) = 3e^t$ then the function $f+g$ is defined by the rule

$$(f+g)(t) = 1 + 2\sin(3t) + 3e^t$$

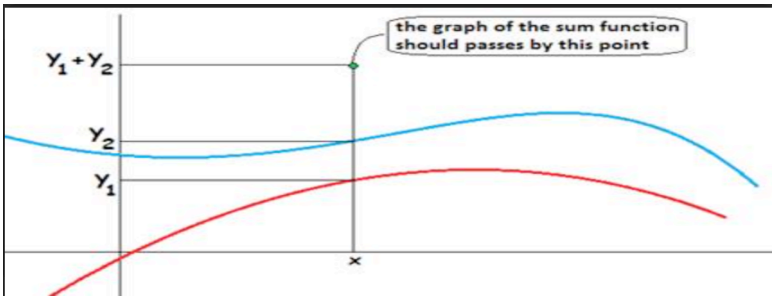
and $7f$ is the function defined by the rule

$$(7f)(t) = 7(1 + 2\sin(3t)) = 7 + 14\sin(3t).$$

Why do the vector space axioms hold? What is the zero vector in this vector space? What is the additive inverse function?

warmup.

One can visualize functions in terms of their graphs - just like for the discrete time signals - and then the graphs of the sum of two functions or of a scalar multiple of one function, are contructed as you'd expect:



short for sub vector space

Definition: A *subspace* of a vector space $V$ is a subset $H$ of $V$ which is itself a vector space with respect to the addition and scalar multiplication in $V$. As soon as one verifies a), b), c) below for $H$, it will be a subspace, because $H$ will "inherit" the other axioms just by being contained in $V$.

a) The zero vector of $V$ is in $H$ (implied by (c), since if $c=0$, $0\vec{u} = \vec{0}$ by (11).

b) $H$ is closed under vector addition, i.e. for each $\underline{\boldsymbol{u}} \in H$, $\underline{\boldsymbol{v}} \in H$ then $\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}} \in H$.

c) $H$ is closed under scalar multiplication, i.e for each $\underline{\boldsymbol{u}} \in H$, $c \in \mathbb{R}$, then also $c\,\underline{\boldsymbol{u}} \in H$.

Just to double check that the other properties get inherited:

Definition A *vector space* is a nonempty set $V$ of objects, called *vectors*, on which are defined two operations, called *addition* and *scalar multiplication,* so that the ten axioms listed below hold. These axioms must hold for all vectors $\underline{\boldsymbol{u}}$, $\underline{\boldsymbol{v}}$, $\underline{\boldsymbol{w}}$ in $V$, and for all scalars $c$, $d \in \mathbb{R}$.

1. The sum of $\underline{\boldsymbol{u}}$ and $\underline{\boldsymbol{v}}$, denoted by $\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}}$, is (also) in $V$ (closure under addition.) inherited

2. $(\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}} = \underline{\boldsymbol{v}} + \underline{\boldsymbol{u}}$ (commutative property of addition) inherited

3. $(\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}}) + \underline{\boldsymbol{w}} = \underline{\boldsymbol{u}} + (\underline{\boldsymbol{v}} + \underline{\boldsymbol{w}})$ (associative property of addition) inherited.

4. There is a **zero** vector $\underline{\mathbf{0}}$ in $V$ so that $\underline{\boldsymbol{u}} + \underline{\mathbf{0}} = \underline{\boldsymbol{u}}$. (additive identity) required (a).

5. For each $\underline{\boldsymbol{u}} \in V$ there is a vector $-\underline{\boldsymbol{u}} \in V$ so that $\underline{\boldsymbol{u}} + (-\underline{\boldsymbol{u}}) = \underline{\mathbf{0}}$. (additive inverses) follows from (c) because $-\vec{u} = (-1)\vec{u}$ (13)

6. The scalar multiple of $u$ by $c$, denoted by $c\,u$ is (also) in $V$. (closure under scalar multiplication) required (c)

7. $c(\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}}) = c\,\underline{\boldsymbol{u}} + c\,\underline{\boldsymbol{v}}$ (scalar multiplication distributes over vector addition) inherited

8. $(c + d)\,\underline{\boldsymbol{u}} = c\,\underline{\boldsymbol{u}} + d\,\underline{\boldsymbol{u}}$. (scalar multiplication distributes over scalar addition) inherited

9. $c\,(d\,\underline{\boldsymbol{u}}) = (c\,d)\,\underline{\boldsymbol{u}}$ (associative property of scalar multiplication) inherited

10. $1\,\underline{\boldsymbol{u}} = \underline{\boldsymbol{u}}$ (multiplicative identity) inherited.

The following three algebra rules follow from the first 10, and are also useful:

11) $0\,\underline{\boldsymbol{u}} = \underline{\boldsymbol{u}}$.

12) $c\,\underline{\mathbf{0}} = \underline{\mathbf{0}}$.

13) $-\underline{\boldsymbol{u}} = (-1)\,\underline{\boldsymbol{u}}$.

} inherited.

There aren't very many sorts of subspaces (sub vector spaces) in $\mathbb{R}^n$.

Big Exercise: (We'll probably start this on Monday, and finish on Tuesday.) The vector space $\mathbb{R}^n$ has subspaces! But there aren't very many kinds, it turns out. (Even though there are countless kinds of *subsets* of $\mathbb{R}^n$.) Let's find *all* the possible kinds of subspaces of $\mathbb{R}^3$, using our expertise with matrix reduced row echelon form.

collect subspaces from small to large in $\mathbb{R}^3$

Let $W$ in $\mathbb{R}^3$ be a sub vector space

(0) $W$ must contain $\vec{0}$

$\{\vec{0}\}$ is a subspace ($\vec{0}+\vec{0}=\vec{0}$, $c\vec{0}=\vec{0}$)

if $W$ contains more,

Let $\vec{u} \in W$, $\vec{u} \neq \vec{0}$.

(1) (c) $\Rightarrow$ $\text{span}\{\vec{u}\}$ contained $W$ by (c).

Note $\text{span}\{\vec{u}\}$ is a subspace:

(b) $c_1\vec{u} + c_2\vec{u} = (c_1+c_2)\vec{u}$

(c) $c(c_1\vec{u}) = (cc_1)\vec{u}$

$W$ could be $\text{span}\{\vec{u}\}$, i.e. a line thru $\vec{0}$.

(2) If $W$ contains more than just $\text{span}\{\vec{u}\}$.

Let $\vec{v} \in W$, $\vec{v} \notin \text{span}\{\vec{u}\}$

by (d) $\text{span}\{\vec{u},\vec{v}\}$ is contained $W$

and $\text{span}\{\vec{u},\vec{v}\}$ is a subspace:

(b) $(c_1\vec{u}+c_2\vec{v}) + (d_1\vec{u}+d_2\vec{v}) = (c_1+d_1)\vec{u} + (c_2+d_2)\vec{v}$

(c) $c(c_1\vec{u}+c_2\vec{v}) = cc_1\vec{u} + cc_2\vec{v}$

$W$ could be a plane thru origin

$$\begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \\ u_3 & v_3 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & * \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \leftarrow \text{can't happen because } \vec{v} \text{ is not multiple of } \vec{u}$$

to be continued...

(3) If $W$ contains more than $\text{span}\{\vec{u},\vec{v}\}$

Let $\vec{w} \in W$, $\vec{w} \notin \text{span}\{\vec{u},\vec{v}\}$.

by (e), $\text{span}\{\vec{u},\vec{v},\vec{w}\}$ is in $W$

and $\text{span}\{\vec{u},\vec{v},\vec{w}\}$ is itself a subspace

b) $(c_1\vec{u}+c_2\vec{v}+c_3\vec{w}) + (d_1\vec{u}+d_2\vec{v}+d_3\vec{w}) = (c_1+d_1)\vec{u} + (c_2+d_2)\vec{v} + (c_3+d_3)\vec{w}$

(note, I'm not thinking about it, but actually using vector space axioms) ■

c) $c(c_1\vec{u}+c_2\vec{v}+c_3\vec{w}) = cc_1\vec{u} + cc_2\vec{v} + cc_3\vec{w}$ ■

Subspaces $W$

(a) $\vec{0} \in W$

(b) $f,g \in W \Rightarrow f+g \in W$

(c) $f \in W$, $c \in \mathbb{R} \Rightarrow cf \in W$

implies

(d) $f,g \in W$, $c_1,c_2 \in \mathbb{R}$

$\Rightarrow c_1f + c_2g \in W$

(use (c) then (d)).

(e) $f,g,h \in W$, $c_1,c_2,c_3 \in \mathbb{R}$

$\Rightarrow (c_1f+c_2g)+c_3h$

from (d), (b), (c)

...

And what is $\text{span}\{\vec{u}, \vec{v}, \vec{w}\}$ ?

Has to be all of $\mathbb{R}^3$ :

algebra!!

$$\begin{bmatrix} \vec{u} & \vec{v} & \vec{w} \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

means $\text{span}\{\vec{u}, \vec{v}, \vec{w}\} = \mathbb{R}^3$ (why?)

$$\begin{bmatrix} 1 & 0 & z_1 \\ 0 & 1 & z_2 \\ 0 & 0 & 0 \end{bmatrix}$$

would mean

$\vec{w} = z_1\vec{u} + z_2\vec{v}$

but $\vec{w}$ not in $\text{span}\{\vec{u}, \vec{v}\}$!

Tues Oct 2

- 4.1-4.2 Vector spaces and subspaces; null spaces, column spaces.

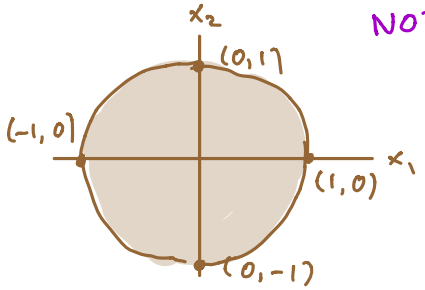
Announcements: part of HW for 10/17:
- 4.1 (1) (3) (5) (7) (13) (19) (31) assigned yesterday
- 4.2 1, (3) (7) (9) (15) (17) (21) (25)

Quiz tomorrow on material thru Wed

Finish big exercise from yesterday, continue with today's.

Warm-up Exercise: Which of these two sets is a subspace of $\mathbb{R}^2$. Why or why not?

a) $\{(x_1, x_2) \mid x_1^2 + x_2^2 \leq 1\} = W_1$




NOT A SUBSPACE

a) TRUE

b) FALSE

just need one counterexample:

$\vec{u} = \begin{bmatrix}1\\0\end{bmatrix}, \vec{v} \in \begin{bmatrix}0\\1\end{bmatrix}$ are in W

$\vec{u} + \vec{v} = \begin{bmatrix}1\\1\end{bmatrix}$ not in W ■

Recall, a subspace W is a sub vector space of the larger vector space ($\mathbb{R}^2$ in this case) so to be a subspace we need b), c) to hold:

a) $\vec{0} \in W$

b) $\vec{u}, \vec{v} \in W \Rightarrow \vec{u} + \vec{v} \in W$

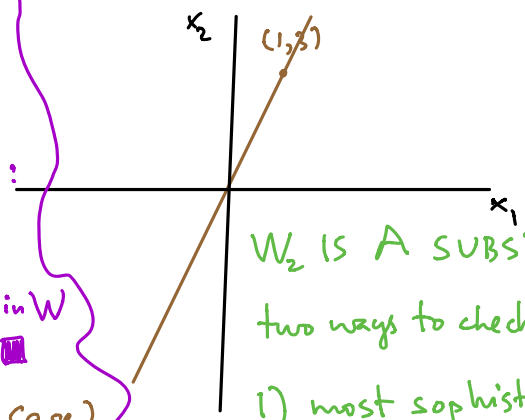
c) $\vec{u} \in W, c \in \mathbb{R} \Rightarrow c\vec{u} \in W$

d) $\vec{u}, \vec{v} \in W, c_1, c_2 \in \mathbb{R} \Rightarrow c_1\vec{u} + c_2\vec{v} \in W$

⋮

(c) FALSE TOO by

b) $\{(x_1, x_2) \mid x_2 = 3x_1\} = W_2$




$W_2$ IS A SUBSPACE!

two ways to check

1) most sophisticated way:

write $W_2 = \text{span}\{\vec{u}\}$

$= \text{span}\left\{\begin{bmatrix}1\\3\end{bmatrix}\right\}$

$\left(\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}x_1\\3x_1\end{bmatrix} = x_1\begin{bmatrix}1\\3\end{bmatrix}\right)$

2) Or, verify (b), (c):

Let $\vec{u} = \begin{bmatrix}a_1\\3a_1\end{bmatrix}, \vec{v} = \begin{bmatrix}b_1\\3b_1\end{bmatrix}$ be in $W_2$

(b) $\vec{u} + \vec{v} \in W_2$?

$= \begin{bmatrix}a_1+b_1\\3a_1+3b_1\end{bmatrix}$ satisfies $x_2 = 3x_1$ so $\vec{u}+\vec{v} \in W$

(c) $c\vec{u} = c\begin{bmatrix}a_1\\3a_1\end{bmatrix} = \begin{bmatrix}ca_1\\c3a_1\end{bmatrix}$ also satisfies $x_2 = 3x_1$

So $c\vec{u} \in W_2$

$W_2$ is a subspace

Finish the big exercise from Monday.

Review....
We've been discussing the abstract notions of *vector spaces* and *subspaces,* with some specific examples to help us with our intuition. Today we continue that discussion. We'll continue to use exactly the same language we used in Chapters 1-2 .... except now it's for general vector spaces:

Let $V$ be a *vector space* (Do you recall that definition, at least roughly speaking?)

Definition: If we have a collection of $p$ vectors $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_p\}$ in $V$, then any vector $\underline{\boldsymbol{v}} \in V$ that can be expressed as a sum of scalar multiples of these vectors is called a *linear combination* of them. In other words, if we can write

$$\underline{\boldsymbol{v}} = c_1\underline{\boldsymbol{v}}_1 + c_2\underline{\boldsymbol{v}}_2 + \ldots + c_p\underline{\boldsymbol{v}}_p ,$$

then $\underline{\boldsymbol{v}}$ is a *linear combination* of $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_p$ . The scalars $c_1, c_2, \ldots, c_p$ are called the *linear combination coefficients* or *weights*.

Definition The *span* of a collection of vectors, written as $span\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_p\}$ , is the collection of all linear combinations of those vectors.

Definition:
a) An indexed set of vectors $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_p\}$ in $V$ is said to be *linearly independent* if no one of the vectors is a linear combination of (some) of the other vectors. The concise way to say this is that the only way $\underline{\mathbf{0}}$ can be expressed as a linear combination of these vectors,

$$c_1\underline{\boldsymbol{v}}_1 + c_2\underline{\boldsymbol{v}}_2 + \ldots + c_p\underline{\boldsymbol{v}}_p = \underline{\mathbf{0}} ,$$

is for all of the weights $c_1 = c_2 = \ldots = c_p = 0$ .

b) An indexed set of vectors $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_p\}$ is said to be *linearly dependent* if at least one of these vectors is a linear combination of (some) of the other vectors. The concise way to say this is that there *is* some way to write $\underline{\mathbf{0}}$ as a linear combination of these vectors

$$c_1\underline{\boldsymbol{v}}_1 + c_2\underline{\boldsymbol{v}}_2 + \ldots + c_p\underline{\boldsymbol{v}}_p = \underline{\mathbf{0}}$$

where *not all* of the $c_j = 0$ . (We call such an equation a *linear dependency*. Note that if we have any such linear dependency, then any $\underline{\boldsymbol{v}}_j$ with $c_j \neq 0$ is a linear combination of the remaining $\underline{\boldsymbol{v}}_k$ with $k \neq j$ . We say that such a $\underline{\boldsymbol{v}}_j$ is *linearly dependent* on the remaining $\underline{\boldsymbol{v}}_k$ .)

And from yesterday,

Definition: A *subspace* of a vector space $V$ is a subset $H$ of $V$ which is itself a vector space with respect to the addition and scalar multiplication in $V$. As soon as one verifies a), b), c) below for $H$, it will be a subspace.

a) The zero vector of $V$ is in $H$

b) $H$ is closed under vector addition, i.e. for each $\underline{\boldsymbol{u}} \in H, \underline{\boldsymbol{v}} \in H$ then $\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}} \in H$.

c) $H$ is closed under scalar multiplication, i.e for each $\underline{\boldsymbol{u}} \in H, c \in \mathbb{R}$, then also $c\,\underline{\boldsymbol{u}} \in H$.

Theorem (spans are subspaces) Let $V$ be a vector space, and let $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n\}$ be a set of vectors in $V$. Then $H = span\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n\}$ is a subspace of $V$.
proof: We need to check that for $H = span\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n\}$

*a) The zero vector of V is in H*

e.g. $0\vec{v}_1 = \vec{0}$ by (13)
$\vec{0} \in H$

*b) H is closed under vector addition, i.e. for each $\underline{\boldsymbol{u}} \in H, \underline{\boldsymbol{v}} \in H$ then $\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}} \in H$.*

Let $\vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$
$\vec{v} = d_1\vec{v}_1 + d_2\vec{v}_2 + \cdots + d_n\vec{v}_n$
Then $\vec{u} + \vec{v} = (c_1+d_1)\vec{v}_1 + (c_2+d_2)\vec{v}_2 + \cdots + (c_n+d_n)\vec{v}_n$.
(after many steps using axioms 1, 2, 3, 6, ...)

*c) H is closed under scalar multiplication, i.e for each $\underline{\boldsymbol{u}} \in H, c \in \mathbb{R}$, then also $c\,\underline{\boldsymbol{u}} \in H$.*

Let $\vec{u}$ as above
$c\vec{u} = cc_1\vec{v}_1 + cc_2\vec{v}_2 + \cdots + cc_n\vec{v}_n$.
(after many steps)

Definition A *basis* of a vector space $V$ is a set of vectors $\{\underline{v}_1, \underline{v}_2, \ldots \underline{v}_n\}$ in $V$ which *spans* $V$ and which is *linearly independent*. (e.g. the standard basis, or other bases in $\mathbb{R}^n$, which we've discussed before.)

Definition A vector space $V$ is *finite dimensional* if it has a basis with only a finite number of vectors in it. Otherwise $V$ is called *infinite dimensional*.

vector space

Definition If $V$ is a finite dimensional then the *dimension* of $V$ is the number of vectors in a basis for $V$. (It turns out that every basis has the same number of vectors, just like every basis of $\mathbb{R}^n$ always has exactly $n$ vectors.

more than $n$ vectors in $\mathbb{R}^n$ are dependent

fewer than $n$ vectors in $\mathbb{R}^n$ can't span $\mathbb{R}^n$.

Remark Using minimal spanning sets, i.e. bases, was how we were able to characterize all possible subspace of $\mathbb{R}^3$ yesterday (or today, if we didn't finish on Monday). Can you characterize all possible ~~subsets~~ subspaces of $\mathbb{R}^n$ in this way?

(0) $\{\vec{0}\}$ has dimension 0

(1) $\text{span}\{\vec{u}\}$ $\vec{u} \neq \vec{0}$ line thru origin dimension = 1

(2) plane thru origin, i.e. $\text{span}\{\vec{u}, \vec{v}\}$ $\vec{u}, \vec{v}$ ind. dim = 2

(3) $\dim(\mathbb{R}^3) = 3$

Example: Let $P_n$ be the space of polynomials of degree at most $n$,

$$P_n = \{p(t) = a_0 + a_1 t + a_2 t^2 + \ldots + a_n t^n \text{ such that } a_0, a_1, \ldots a_n \in \mathbb{R}\}$$

Note that $P_n$ is the span of the $(n+1)$ functions

$$p_0(t) = 1, p_1(t) = t, p_2(t) = t^2, \ldots\ p_n(t) = t^n.$$

Although we often consider $P_n$ as a vector space on its own, we can also consider it to be a subspace of the much larger vector space $V$ of all functions from $\mathbb{R}$ to $\mathbb{R}$.

(recall: a basis for V is a set of vectors that spans V & is linearly independent) (domain: $-\infty < t < \infty$)

Exercise 1 abbreviating the functions by their formulas, we have

$$P_3 = span\{1, t, t^2, t^3\}. \quad = \{p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3,\ a_0, a_1, a_2, a_3 \in \mathbb{R}\}$$

Are the functions in the set $\{1, t, t^2, t^3\}$ linearly independent or linearly dependent? Are they a basis for $P_3$?

- By construction these functions span $P_3$.
- Think if you can show they're linearly independent

to be continued...

$*$ $c_1 \cdot 1 + c_2 t + c_3 t^2 + c_4 t^3 \equiv 0$ ← the zero fcn, which is 0 for all $t$

$\Rightarrow c_1 = c_2 = c_3 = c_4 = 0$?

- One way: pick some good $t$'s.

@ $t = 0$ $*$ $\Rightarrow c_1 + 0 + 0 + 0 = 0 \Rightarrow c_1 = 0.$

$\Rightarrow c_2 t + c_3 t^2 + c_4 t^3 = 0$

@ $t = 1$ $\Rightarrow c_2 + c_3 + c_4 = 0$
@ $t = -1$ $\Rightarrow -c_2 + c_3 - c_4 = 0$
$E_1 + E_2 \Rightarrow 2c_3 = 0. \Rightarrow c_3 = 0$

$\Rightarrow c_2 t + c_4 t^3 = 0$

@ $t = 1$ $c_2 + c_4 = 0$
@ $t = 2$ $2c_2 + 8c_4 = 0$

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 2 & 8 & 0 \end{array}\right] \quad -2R_1 + R_2 \to R_2 \quad \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 6 & 0 \end{array}\right] \Rightarrow c_4 = 0,\ c_2 = 0$$

(It turns out if you had used any 4 different $t$ values would've gotten a homog. sys. of 4 eqns in 4 unknowns & only soln would've been $c_1 = c_2 = c_3 = c_4 = 0$

Better: use Calculus: $c_1 + c_2 t + c_3 t^2 + c_4 t^3 \equiv 0 \Rightarrow c_1 = 0$

$D_t$: $c_2 + 2tc_3 + 3c_4 t^2 \equiv 0 \Rightarrow c_2 = 0$

$D_t$: $2c_3 + 6c_4 t \equiv 0 \Rightarrow c_3 = 0$

$D_t$: $6c_4 \equiv 0 \Rightarrow c_4$

We've seen that one (explicit) way that subspaces arise is as the span of a specified collection of vectors. The primary (implicit) way that subspaces are described is related to the following:

"zero" H $\quad \text{Nul } A = \{\vec{x} \in \mathbb{R}^n \text{ s.t. } A\vec{x} = \vec{0}\}$.

Definition: The *null space* of an $m \times n$ matrix $A$ is the set of $\underline{\boldsymbol{x}} \in \mathbb{R}^n$ for which $A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{0}}$. We denote this set by *Nul A*. Equivalently, in terms of the associated linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ given by $T(\underline{\boldsymbol{x}}) = A\,\underline{\boldsymbol{x}}$, *Nul A* is the set of points in the domain which are transformed into the zero vector in the codomain.

Theorem Let $A$ be an $m \times n$ matrix. Then *Nul A* is a subspace of $\mathbb{R}^n$.

proof: We need to check that for $H = Nul(A)$:

*a) The zero vector of V is in H*

is $\vec{0} \in Nul(A)$ $\quad (\vec{0} \in \mathbb{R}^n)$

$A\vec{0} = \vec{0}$ ✓

*b) H is closed under vector addition, i.e. for each* $\underline{\boldsymbol{u}} \in H$, $\underline{\boldsymbol{v}} \in H$ *then* $\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}} \in H$.

so $A\vec{u} = \vec{0}$

$A\vec{v} = \vec{0}$.

so $A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v} = \vec{0} + \vec{0} = \vec{0}$.

so $\vec{u} + \vec{v}$ is a homog. soln.

*c) H is closed under scalar multiplication, i.e for each* $\underline{\boldsymbol{u}} \in H$, $c \in \mathbb{R}$, *then also* $c\,\underline{\boldsymbol{u}} \in H$.

so $A\vec{u} = \vec{0}$

so $Ac\vec{u} = cA\vec{u} = c\vec{0} = \vec{0}$

so $c\vec{u}$ is a homog soln

$\mathbb{R}^5$ $x_1, x_2 \cdots x_5$

Exercise 2a) For the matrix $A$ below, express the vectors in $Nul(A)$ explicitly, using the methods of Chapters 1-2.

$$A = \left[\begin{array}{ccccc|c} 1 & -2 & 0 & -1 & 1 & 0 \\ -2 & 4 & 0 & 2 & -2 & 0 \\ 3 & -6 & 4 & 1 & 7 & 0 \end{array}\right] \text{ reduces to } \left[\begin{array}{ccccc|c} 1 & -2 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 = 2x_2 + x_4 - x_5$
$x_2 = x_2$ free
$x_3 = -x_4 - x_5$
$x_4 = x_4$ free
$x_5 = x_5$ free

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

2b) Exhibit a basis for $Nul(A)$. $= \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ Nul A

- by construction, these 3 vectors span Nul A
- they're also linearly independent:

$$x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ 2nd entries: $x_2 + 0 + 0 = 0$
4th entries $x_4 = 0$
5th entries $x_5 = 0$

- so they're a basis!!

Definition Let $A$ be an $m \times n$ matrix, expressed in column form as $A = [\underline{\boldsymbol{a}}_1\ \underline{\boldsymbol{a}}_2\ \underline{\boldsymbol{a}}_3\ \ldots\ \underline{\boldsymbol{a}}_n]$ The *column space* of $A$, written as *Col A*, is the span of the columns:

$$Col\ A = span\{\underline{\boldsymbol{a}}_1\ \underline{\boldsymbol{a}}_2\ \underline{\boldsymbol{a}}_3\ \ldots\ \underline{\boldsymbol{a}}_n\}.$$

Equivalently, since

$$A\,\underline{\boldsymbol{x}} = x_1\,\underline{\boldsymbol{a}}_1 + x_2\,\underline{\boldsymbol{a}}_2 + \ldots + x_n\,\underline{\boldsymbol{a}}_n$$

we see that *Col A* is also the range of the linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ given by $T(\underline{\boldsymbol{x}}) = A\,\underline{\boldsymbol{x}}$, i.e

$$Col\ A = \{\underline{\boldsymbol{b}} \in \mathbb{R}^m \text{ such that } \underline{\boldsymbol{b}} = A\,\underline{\boldsymbol{x}} \text{ for some } \underline{\boldsymbol{x}} \in \mathbb{R}^n\}.$$

Theorem By the "spans are subspaces" theorem, $Col(A)$ is always a subspace of $\mathbb{R}^m$.

Tuesday notes

pick up your graded exams!

Exercise 3a) Consider the same matrix we considered in Exercise 2

$$A = \begin{bmatrix} 1 & -2 & 0 & -1 & 1 \\ -2 & 4 & 0 & 2 & -2 \\ 3 & -6 & 4 & 1 & 7 \end{bmatrix}$$
$$\vec{a}_1 \quad \vec{a}_2 \quad \vec{a}_3 \quad \vec{a}_4 \quad \vec{a}_5$$

at least 2 independent vectors

By the Theorem, $col(A)$ is a subspace of $\mathbb{R}^3$. Which is it: ~~{0}~~, a line thru the origin, a plane thru the origin, or all ~~of~~ $\mathbb{R}^3$. Hint:

there are non-zero columns.

$A\vec{x} = \vec{b}$

Sometimes inconsistent since rref(A) has a zero row

$$\begin{bmatrix} 1 & -2 & 0 & -1 & 1 \\ -2 & 4 & 0 & 2 & -2 \\ 3 & -6 & 4 & 1 & 7 \end{bmatrix} \quad \text{reduces to} \quad \begin{bmatrix} 1 & -2 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$
$$\vec{a}_1 \quad \vec{a}_2 \quad \vec{a}_3 \quad \vec{a}_4 \quad \vec{a}_5$$

Friday warmup: Express $\vec{a}_2, \vec{a}_4, \vec{a}_5$ in terms of $\vec{a}_1, \vec{a}_3$. Also try to answer 3a

$\vec{a}_2 = -2\vec{a}_1$ ; $\vec{a}_4 = -\vec{a}_1 + \vec{a}_3$ $\quad \vec{a}_5 = \vec{a}_1 + \vec{a}_3$

(look at rref(A)) (since true for rref)

3b) Is there a more efficient way to express *Col A* as a span that doesn't require all five column vectors?

Yes. In fact, $ColA = span\{\vec{a}_1, \vec{a}_3\}$

Why: $span\{\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4, \vec{a}_5\}$ = set of all linear combos of these 5 vectors.

$= span\{\vec{a}_1, \vec{a}_3\}$.

$$c_1\vec{a}_1 + c_2\vec{a}_2 + c_3\vec{a}_3 + c_4\vec{a}_4 + c_5\vec{a}_5$$

$$= c_1\vec{a}_1 + c_2(-2\vec{a}_1) + c_3\vec{a}_3 + c_4(-\vec{a}_1 + \vec{a}_3) + c_5(\vec{a}_1 + \vec{a}_3)$$

$$= (c_1 - 2c_2 - c_4 + c_5)\vec{a}_1 + (c_3 + c_4 + c_5)\vec{a}_3$$

$$= d_1\vec{a}_1 + d_3\vec{a}_3$$

Also $\{\vec{a}_1, \vec{a}_3\}$ are linearly independent (because corresponding columns in rref(A) are independent)

3c Exhibit a basis for *col A* in Exercise 2.

$\{\vec{a}_1, \vec{a}_3\} = \left\{ \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} \right\}$

verifies col A is plane thru origin

Wed Oct 3

- 4.2 - 4.3 nullspaces and column spaces; kernel and range of linear transformations as subspaces. Linearly independent sets and bases for vector spaces.

Announcements:
- Hw for Oct 17 is posted.
- exams graded by Friday
- Today, Col A, Nul A for $m \times n$ matrices A

(i.e. finish Tuesday's notes) ↑ span of the columns of A ↑ $\{$solutions $\vec{x}$ to $A\vec{x} = \vec{0}\}$

'til 12:58

Exercise 1 Tuesday.

(This is where we ended yesterday, do the exercise there.)

Warm-up Exercise: We ended class yesterday thinking about how to show that $p_0(t) = 1$, $p_1(t) = t$, $p_2(t) = t^2$ and $p_3(t) = t^3$ are a basis for

$$\mathbb{P}_3 := \{ p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 \mid a_0, a_1, a_2, a_3 \in \mathbb{R} \}$$

i.e. the subspace of polynomials $p(t)$ of degree $\leq 3$.

(We assume the domain of our functions is $-\infty < t < \infty$, i.e. $\mathbb{R}$.)

It was clear that span $\{p_0, p_1, p_2, p_3\} = \mathbb{P}_3$, (which is why we know $\mathbb{P}_3$ is a subspace)

so it remained to show that

$\{p_0, p_1, p_2, p_3\}$ is linearly independent.

Try to show this by beginning with the dependency equation

$$c_0 + c_1 t + c_2 t^2 + c_3 t^3 = 0$$ ← the zero function, which is the zero "vector" in the space of functions $f: \mathbb{R} \to \mathbb{R}$.

and trying different $t$-values to show

$c_0 = c_1 = c_2 = c_3 = 0$.

(This is Exercise 1 in Tuesday notes)

The ideas of nullspace and column space generalize to arbitrary linear transformations between vectors spaces - with slightly more general terminology.

Definition Let $V$ and $W$ be vector spaces. A function $T : V \to W$ is called a *linear transformation* if for each $\underline{\boldsymbol{x}} \in V$ there is a unique vector $T(\underline{\boldsymbol{x}}) \in W$ and so that

(i) $T(\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}}) = T(\underline{\boldsymbol{u}}) + T(\underline{\boldsymbol{v}})$ for all $\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}} \in V$

(ii) $T(c\,\underline{\boldsymbol{u}}) = c\,T(\underline{\boldsymbol{u}})$ for all $\underline{\boldsymbol{u}} \in V, c \in \mathbb{R}$

Definition The *kernel* (or *nullspace)* of $T$ is defined to be $\{\underline{\boldsymbol{u}} \in V : T(\underline{\boldsymbol{u}}) = \underline{\mathbf{0}}\}$.

Definition The *range* of $T$ is $\{\underline{\boldsymbol{w}} \in W : \underline{\boldsymbol{w}} = T(\underline{\boldsymbol{v}})$ *for some* $\underline{\boldsymbol{v}} \in V\}$ .

Theorem Let $T : V \to W$ be a *linear transformation.* Then the kernel of $T$ is a subspace of $V$. The range of $T$ is a subspace of $W$.

Remark: The theorem generalizes our earlier one about *Nul A* and *Col A*, for matrix transformations $T : \mathbb{R}^n \to \mathbb{R}^m$, $T(\underline{\boldsymbol{x}}) = A\,\underline{\boldsymbol{x}}$.

$= C^1[a,b]$

Exercise 1 Let $V$ be the vector space of real-valued functions $f$ defined on an interval $[a, b]$ with the property that they are differentiable and that their derivatives are continuous functions on $[a, b]$. Let $W$ be the vector space $C[a, b]$ of all continous functions on the interval $[a, b]$. Let $D : V \to W$ be the derivative transformation

$$D(f) = f' .$$

1a) What Calculus differentiation rules tell you that D is a linear transformation?

D is linear : Let $f, g \in C^1[a,b] = V$

$D(f+g) = (f+g)' = f' + g'$

$c \in \mathbb{R}$ : $D(cf) = (cf)' = cf'$

(a) $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$

(b) $T(c\vec{u}) = cT(\vec{u})$

1b) What subspace is the kernel of D ?

You know ~~fo~~ from calculus that $f'(x) \equiv 0$ (identically ↓ equal to) on $[a,b]$ if and only if $f(x) \equiv c \in \mathbb{R}$

Explicit way. $\ker D = \text{span}\{f_1(x)\}$ $f_1(x) \equiv 1$

every constant fcn is $c \cdot f_1(x) = c \cdot 1$ for some $c$

2c) What is the range of D ?

all of $C[a,b] = W$.

Part I of FTC.

We proved in Calc. that if $g(x)$ is continuous on $[a,b]$ then $\int_a^x g(t)\,dt := f(x)$ is an antiderivative

i.e. $Df = g$.

Not all bases are created equal!

Theorem: Let $span\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_p\} = H$ be a subspace. The following *elementary operations* do not effect the span of the resulting ordered set:

(i) swap two of the vectors in the set, i.e. replace $\underline{\boldsymbol{v}}_j$ with $\underline{\boldsymbol{v}}_k$, and replace $\underline{\boldsymbol{v}}_k$ with $\underline{\boldsymbol{v}}_j$ .

(ii) replace $\underline{\boldsymbol{v}}_j$ with $c\,\underline{\boldsymbol{v}}_j$, for $c \neq 0$.

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_j\vec{v}_j + \cdots + c_p\vec{v}_p = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots \frac{c_j}{c}(c\vec{v}_j) + \cdots + c_p\vec{v}_p$$

(iii) for $j \neq k$, replace $\underline{\boldsymbol{v}}_k$ with $\underline{\boldsymbol{v}}_k + c\,\underline{\boldsymbol{v}}_j$ .

$$c_1\vec{v}_1 + \cdots + c_k\vec{v}_k + \cdots + c_j\vec{v}_j + \cdots + c_p\vec{v}_p = c_1\vec{v}_1 + \cdots c_k(\vec{v}_k + c\vec{v}_j) + \cdots + (c_j - cc_k)\vec{v}_j + \cdots + c_p\vec{v}_p$$

Exercise 2) Use the "change of spanning set" theorem above, to find a better basis for *Col A* then the one we came up with by culling dependent vectors, on Friday. Hint: Use elementary column operations to compute the reduced column echelon form of *A*. Illustrate why this new basis is a better basis for *Col A* by seeing how easy it is to express any one of the original column vectors in terms of this improved basis.

In this example, $A = [\underline{\boldsymbol{a}}_1\ \underline{\boldsymbol{a}}_2\ \underline{\boldsymbol{a}}_3\ \underline{\boldsymbol{a}}_4\ \underline{\boldsymbol{a}}_5]$ and $Col\,A = span\{\underline{\boldsymbol{a}}_1, \underline{\boldsymbol{a}}_2, \underline{\boldsymbol{a}}_3, \underline{\boldsymbol{a}}_4, \underline{\boldsymbol{a}}_5\}$

$$A = \begin{bmatrix} 1 & -2 & 0 & -1 & 1 \\ -2 & 4 & 0 & 2 & -2 \\ 3 & -6 & 4 & 1 & 7 \end{bmatrix}.$$

As we just reviewed, a pretty good basis for *Col A* is $\{\underline{\boldsymbol{a}}_1, \underline{\boldsymbol{a}}_3\}$:

$$\begin{bmatrix} 1 & -2 & 0 & -1 & 1 \\ -2 & 4 & 0 & 2 & -2 \\ 3 & -6 & 4 & 1 & 7 \end{bmatrix} \quad \text{row reduces to} \quad \begin{bmatrix} 1 & -2 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Now column reduce $A$ to get a basis for $Col\ A$ that's as good as you could hope for....and show this by expressing each of the original columns in terms of this basis.

$$A = \begin{bmatrix} 1 & -2 & 0 & -1 & 1 \\ -2 & 4 & 0 & 2 & -2 \\ 3 & -6 & 4 & 1 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 \\ 3 & 0 & 4 & 4 & 4 \end{bmatrix}$$

$2\vec{a}_1 + \vec{a}_2 \to \vec{a}_2$ $\quad -\vec{a}_1 + \vec{a}_5 \to \vec{a}_5$

$\vec{a}_1 + \vec{a}_4 \to \vec{a}_4$

$$\to \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 \\ 3 & 4 & 4 & 4 & 0 \end{bmatrix}$$

$\vec{a}_3 \to \vec{a}_3$ $\quad \vec{a}_5 \to \vec{a}_5$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$\vec{a}_2 \to \vec{a}_2/4$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

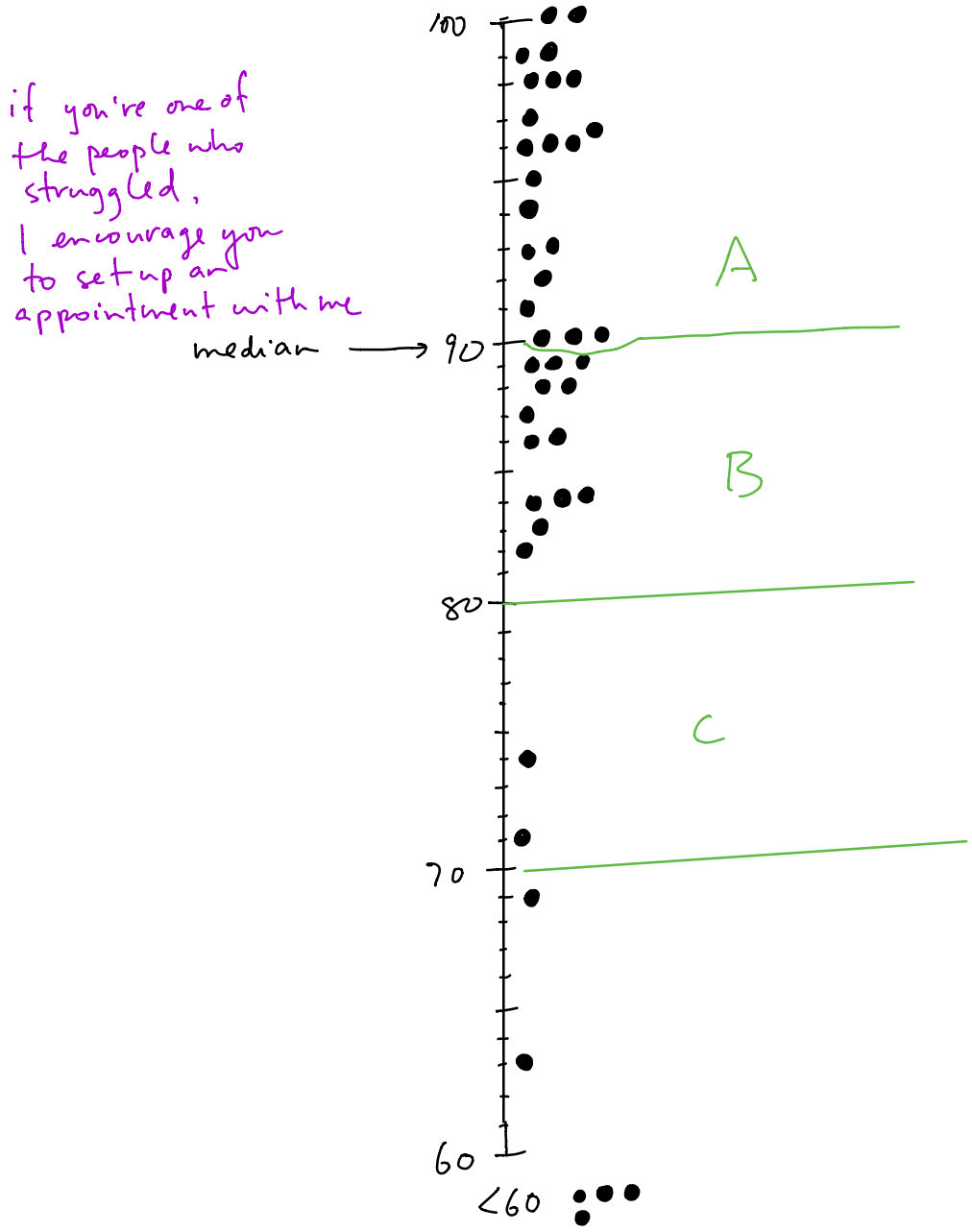
$-3\vec{a}_2 + \vec{a}_1 \to \vec{a}_1$

very good basis for Col A is

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Exam 1 Scores n = 43

2270-2 Fall 18




if you're one of the people who struggled, I encourage you to set up an appointment with me

Fri Oct 4

- 4.4 Coordinate systems for finite dimensional vector spaces

Announcements:

- overall, the class did great on exam 1 (contact me if otherwise for you).
- the current material is more difficult.
- We'll discuss remaining "Tuesday" "Wednesday" notes today. (Friday material after break)

Warm-up Exercise: See column space discussion in "Tuesday" notes