Math 2270-002 Week 6 notes
We will not necessarily finish the material from a given day's notes on that day. We may also add or subtract some material as the week progresses, but these notes represent an in-depth outline of what we plan to cover. These notes cover material in 3.1-3.3, as well as some concepts review for our Friday midterm, in the Wednesday notes. Our exam on Friday includes this material.

Mon Sept 24

- 3.2 properties of determinants

Announcements:

Warm-up Exercise:

recall from Friday that

Definition: Let $A_{n \times n} = [a_{ij}]$. Then the determinant of A, written $det(A)$ or $|A|$, is defined by

$$det(A) := \sum_{j=1}^{n} a_{1j}(-1)^{1+j}M_{1j} = \sum_{j=1}^{n} a_{1j}C_{1j}.$$

Here M_{1j} is the determinant of the $(n-1) \times (n-1)$ matrix obtained from A by deleting the first row and the j^{th} column, and C_{1j} is simply $(-1)^{1+j}M_{1j}$.

More generally, the determinant of the $(n-1) \times (n-1)$ matrix obtained by deleting row i and column j from A is called the $i\,j$ Minor M_{ij} of A, and $C_{ij} := (-1)^{i+j}M_{ij}$ is called the $i\,j$ Cofactor of A.

Theorem: $det(A)$ can be computed by expanding across any row, say row i:

$$det(A) := \sum_{j=1}^{n} a_{ij}(-1)^{i+j}M_{ij} = \sum_{j=1}^{n} a_{ij}C_{ij}$$

or by expanding down any column, say column j:

$$det(A) := \sum_{i=1}^{n} a_{ij}(-1)^{i+j}M_{ij} = \sum_{i=1}^{n} a_{ij}C_{ij}.$$

(proof is not so easy - our text skips it and so will we. If you look on Wikipedia and as we illustrated for 3×3 matrices on Friday, the determinant is actually a sum of n factorial terms, each of which is $\pm$ a product of n entries of A where each product has exactly one entry from each row and column. The $\pm$ sign has to do with whether the corresponding permutation is even or odd. You can verify this pretty easily for the 2×2 and 3×3 cases. One can show, using math induction, that each row or column cofactor expansion above reproduces this sum, in the $n \times n$ case.)

We also illsutrated the following theorem, which we will understand the reasons for partly today, and partly on Tuesday:

Theorem Let A be a square matrix. Then A^{-1} exists if an only if $|A| \neq 0$. And in this case

$$A^{-1} = \frac{1}{|A|}\, C^T$$

where $C = [C_{ij}]$ is the matrix of cofactors of A. (The matrix C^T is called the *"adjoint matrix"* in most texts, although ours prefers to use the word "*adjugate*", because "adjoint" has another meaning as well in linear algebra.)

Exercise 1) Compute the following determinants by being clever about which rows or columns to use:

1a) $\begin{vmatrix} 1 & 38 & 106 & 3 \\ 0 & 2 & 92 & -72 \\ 0 & 0 & 3 & 45 \\ 0 & 0 & 0 & -2 \end{vmatrix}$; 1b) $\begin{vmatrix} 1 & 0 & 0 & 0 \\ \pi^2 & 2 & 0 & 0 \\ 0.476 & 88 & 3 & 0 \\ 1 & 22 & 33 & -2 \end{vmatrix}$.

Exercise 2) Explain why it is always true that for an upper triangular matrix (as in 1a), or for a lower triangular matrix (as in 1b), the determinant is always just the product of the diagonal entries.

The effective way to compute determinants for larger-sized matrices without lots of zeroes is to not use the definition directly, but rather to use elementary row operations to make the matrix upper triangular, along with the following facts, which track how elementary row operations affect determinants.

- (1a) Swapping any two rows of a matrix changes the sign of the determinant.
  proof: This is clear for 2×2 matrices, since

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12} \qquad \begin{vmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{vmatrix} = a_{21}a_{12} - a_{11}a_{22}$$

  For 3×3 determinants, expand across the row *not* being swapped, and use the 2×2 swap property to deduce the result. Prove the general result by induction: once it's true for $n \times n$ matrices you can prove it for any $(n+1) \times (n+1)$ matrix, by expanding across a row that wasn't swapped, and applying the $n \times n$ result.

(1b) Thus, if two rows in a matrix are the same, the determinant of the matrix must be zero: on the one hand, swapping those two rows leaves the matrix and its determinant unchanged; on the other hand, by (1a) the determinant changes its sign. The only way this is possible is if the determinant is zero.

- (2a) If you factor a constant out of a row, then you factor the same constant out of the determinant. Precisely, using $\mathcal{R}_i$ for i^{th} row of A, and writing $\mathcal{R}_i = c\,\mathcal{R}_i^*$

$$\begin{vmatrix} \mathcal{R}_1 \\ \mathcal{R}_2 \\ \vdots \\ \mathcal{R}_i \\ \mathcal{R}_n \end{vmatrix} = \begin{vmatrix} \mathcal{R}_1 \\ \mathcal{R}_2 \\ \vdots \\ c\,\mathcal{R}_i^* \\ \mathcal{R}_n \end{vmatrix} = c \begin{vmatrix} \mathcal{R}_1 \\ \mathcal{R}_2 \\ \vdots \\ \mathcal{R}_i^* \\ \mathcal{R}_n \end{vmatrix} .$$

proof: expand across the i^{th} row, noting that the corresponding cofactors don't change, since they're computed by deleting the i^{th} row to get the corresponding minors:

$$det(A) = \sum_{j=1}^{n} a_{ij} C_{ij} = \sum_{j=1}^{n} c\, a_{ij}^* C_{ij} = c \sum_{j=1}^{n} a_{ij}^* C_{ij} = c\, det(A^*) .$$

(2b) Combining (2a) with (1b), we see that if one row in A is a scalar multiple of another, then $det(A) = 0$.

- (3) If you replace row i of A, $\mathcal{R}_i$ by its sum with a multiple of another row, say $\mathcal{R}_k$ then the determinant is unchanged! Expand across the i^{th} row:

$$\begin{vmatrix} \mathcal{R}_1 \\ \mathcal{R}_2 \\ \mathcal{R}_k \\ \mathcal{R}_i + c\,\mathcal{R}_k \\ \mathcal{R}_n \end{vmatrix} = \sum_{j=1}^{n} \left(a_{ij} + c\,a_{kj}\right) C_{ij} = \sum_{j=1}^{n} a_{ij} C_{ij} + c \sum_{j=1}^{n} a_{kj} C_{ij} = det(A) + c \begin{vmatrix} \mathcal{R}_1 \\ \mathcal{R}_2 \\ \mathcal{R}_k \\ \mathcal{R}_k \\ \mathcal{R}_n \end{vmatrix} = det(A) + 0\,.$$

<u>Remark:</u> The analogous properties hold for corresponding "elementary column operations". In fact, the proofs are almost identical, except you use the corresponding column expansions instead of row expansions.

Exercise 3) Recompute $\begin{vmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{vmatrix}$ from Friday (using row and column expansions we always got an answer of 15 then.) This time use elementary row operations (and/or elementary column operations) to reduce the matrix into triangular form first.

Exercise 4) Compute $\begin{vmatrix} 1 & 0 & -1 & 2 \\ 2 & 1 & 1 & 0 \\ 2 & 0 & 1 & 1 \\ -1 & 0 & -2 & 1 \end{vmatrix}$.

Theorem: Let $A_{n \times n}$. Then A^{-1} exists if and only if $det(A) \neq 0$.

proof: We already know that A^{-1} exists if and only if the reduced row echelon form of A is the identity matrix. Now, consider reducing A to its reduced row echelon form, and keep track of how the determinants of the corresponding matrices change: As we do elementary row operations,

- if we swap rows, the sign of the determinant switches.
- if we factor non-zero factors out of rows, we factor the same factors out of the determinants.
- if we replace a row by its sum with a multiple of another row, the determinant is unchanged.

Thus,

$$|A| = c_1|A_1| = c_1c_2|A_2| = \ldots = c_1c_2 \ldots c_N|rref(A)|$$

where the nonzero c_k's arise from the three types of elementary row operations. If $rref(A) = I$ its determinant is 1, and $|A| = c_1c_2 \ldots c_N \neq 0$. If $rref(A) \neq I$ then its bottom row is all zeroes and its determinant is zero, so $|A| = c_1c_2 \ldots c_N(0) = 0$. Thus $|A| \neq 0$ if and only if $rref(A) = I$ if and only if A^{-1} exists !

Theorem: Using the same ideas as above, we can show that $det(A\,B) = det(A)det(B)$. This is an important identity that gets used, for example, in multivariable change of variables formulas for integration, using the Jacobian matrix. (It is not true that $det(A + B) = det(A) + det(B)$.)

Here's how to show $det(A\,B) = det(A)det(B)$: The key point is that if you do an elementary row operation to AB, that's the same as doing the elementary row operation to A, and then multiplying by B. With that in mind, if you do exactly the same elementary row operations as you did for A in the theorem above, you get

$$|A\,B| = c_1|A_1B| = c_1c_2|A_2B| = \ldots = c_1c_2 \ldots c_N|rref(A)B|.$$

If $rref(A) = I$, then from the theorem above, $|A| = c_1c_2 \ldots c_N$, and we deduce $|A\,B| = |A||B|$. If $rref(A) \neq I$, then its bottom row is zeroes, and so is the bottom row of $rref(A)B$. Thus $|A\,B| = 0$ and also $|A||B| = 0$.

Tues Sept 25

- 3.3 adjugate formula for inverses, Cramer's rule, geometric meanings of determinants.

Announcements:

Warm-up Exercise:

Theorem: Let $A_{n \times n}$, and denote its cofactor matrix by $cof(A) = [C_{ij}]$, with $C_{ij} = (-1)^{i+j} M_{ij}$, and $M_{ij} =$ the determinant of the $(n-1) \times (n-1)$ matrix obtained by deleting row i and column j from A. Define the adjugate matrix to be the transpose of the cofactor matrix:

$$Adj(A) := cof(A)^T$$

Then, when A^{-1} exists it is given by the formula

$$A^{-1} = \frac{1}{det(A)} Adj(A) .$$

Exercise 1) Show that in the 2 × 2 case this reproduces the formula

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \frac{1}{det(A)} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} .$$

Example) For our friend $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{bmatrix}$ from last Friday we worked out $cof(A) = \begin{bmatrix} 5 & 2 & -6 \\ 0 & 3 & 6 \\ 5 & -1 & 3 \end{bmatrix}$ so

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{bmatrix} \qquad adj(A) = \begin{bmatrix} 5 & 0 & 5 \\ 2 & 3 & -1 \\ -6 & 6 & 3 \end{bmatrix} \qquad det(A) = 15,$$

so

$$A^{-1} = \frac{1}{15}\begin{bmatrix} 5 & 0 & 5 \\ 2 & 3 & -1 \\ -6 & 6 & 3 \end{bmatrix}$$

Let's understand why the magic worked:

Exercise 2) Continuing with our example,

2a) The $(1, 1)$ entry of $(A)(Adj(A))$ is $15 = 1 \cdot 5 + 2 \cdot 2 + (-1)(-6)$. Explain why this is $det(A)$, expanded across the first row.

2b) The $(2, 1)$ entry of $(A)(Adj(A))$ is $0 \cdot 5 + 3 \cdot 2 + (1)(-6) = 0$. Notice that you're using the same cofactors as in (2a). What matrix, which is obtained from A by keeping two of the rows, but replacing a third one with one of those two, is this the determinant of?

2c) The $(3, 2)$ entry of $(A)(Adj(A))$ is $2 \cdot 0 - 2 \cdot 3 + 1 \cdot 6 = 0$. What matrix (which uses two rows of A) is this the determinant of?

If you completely understand 2abc, then you have realized why

$$[A][Adj(A)] = det(A)[I]$$

for every square matrix, and so also why

$$A^{-1} = \frac{1}{det(A)} Adj(A) .$$

Precisely,

$$entry_{i\,i}\, A(Adj(A)) = row_i(A) \cdot col_i(Adj(A)) = row_i(A) \cdot row_i(cof(A)) = det(A),$$

expanded across the i^{th} row.

On the other hand, for $i \neq k$,

$$entry_{k\,i}\, A(Adj(A)) = row_k(A) \cdot col_i(Adj(A)) = row_k(A) \cdot row_i(cof(A)) .$$

This last dot produce is zero because it is the determinant of a matrix made from A by replacing the i^{th} row with the k^{th} row, expanding across the i^{th} row, and whenever two rows are equal, the determinant of a matrix is zero:

$$i^{th} \text{ row position} \begin{vmatrix} \mathcal{R}_1 \\ \mathcal{R}_2 \\ \mathcal{R}_k \\ \mathcal{R}_k \\ \mathcal{R}_n \end{vmatrix} .$$

There's a related formula for solving for individual components of $\underline{\mathbf{x}}$ when $A\,\underline{\mathbf{x}} = \underline{\boldsymbol{b}}$ has a unique solution ($\underline{\mathbf{x}} = A^{-1}\underline{\boldsymbol{b}}$). This can be useful if you only need one or two components of the solution vector, rather than all of it:

Cramer's Rule: Let $\underline{\mathbf{x}}$ solve $A\,\underline{\mathbf{x}} = \underline{\boldsymbol{b}}$, for invertible A. Then

$$x_k = \frac{det(A_k)}{det(A)}$$

where A_k is the matrix obtained from A by replacing the k^{th} column with $\underline{\boldsymbol{b}}$.

proof: Since $\underline{\mathbf{x}} = A^{-1}\underline{\boldsymbol{b}}$ the k^{th} component is given by

$$\begin{aligned} x_k &= entry_k\left(A^{-1}\underline{\boldsymbol{b}}\right) \\ &= entry_k\left(\frac{1}{|A|}Adj(A)\underline{\boldsymbol{b}}\right) \\ &= \frac{1}{|A|}row_k(Adj(A)) \bullet \underline{\boldsymbol{b}} \\ &= \frac{1}{|A|}col_k(cof(A)) \bullet \underline{\boldsymbol{b}}. \end{aligned}$$

Notice that $col_k(cof(A)) \bullet \underline{\boldsymbol{b}}$ is the determinant of the matrix obtained from A by replacing the k^{th} column by $\underline{\boldsymbol{b}}$, where we've computed that determinant by expanding down the k^{th} column! This proves the result. (See our text for another way of justifying Cramer's rule.)

Exercise 3) Solve $\begin{bmatrix} 5 & -1 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$.

3a) With Cramer's rule

3b) With A^{-1}, using the adjoint formula.

Wed Sept 26

- 3.3 geometric meaning of determinants as area/volume expansion factors; review material for Friday exam.

Announcements:

Warm-up Exercise:

Discussion on determinants as area/volume expansion factors for the associated linear transformation

In your last homework problem this week you used sort of an ad-hoc area computation to connect the (absolute value of the) determinant of a 2×2 matrix A with the area expansion factor of the linear transformation $T(\underline{\mathbf{x}}) = A\,\underline{\mathbf{x}}$. There is a more systematic approach which works in dimensions three and higher as well, which is based on the idea that every matrix is a product of elementary matrices. Elementary matrices correspond to elementary row operations, but when considering the associated linear transformations and looking at the columns, they correspond to scaling a single column, interchanging two columns, and shears. For example,

a) The linear transformation of $\mathbb{R}^3$ to itself associated to the elementary matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

scales the x_2 direction by a factor of -3, and leaves the x_1 and x_3 coordinates unchanged. Its determinant, which is -3 has absolute value equal to the volume expansion factor of the transformation.

b) The linear transformation of $\mathbb{R}^3$ to itself associated to the elementary matrix

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

interchanges the $\underline{\boldsymbol{e}}_1$ and $\underline{\boldsymbol{e}}_3$ directions. Its determinant is -1, and its volume expansion factor is 1.

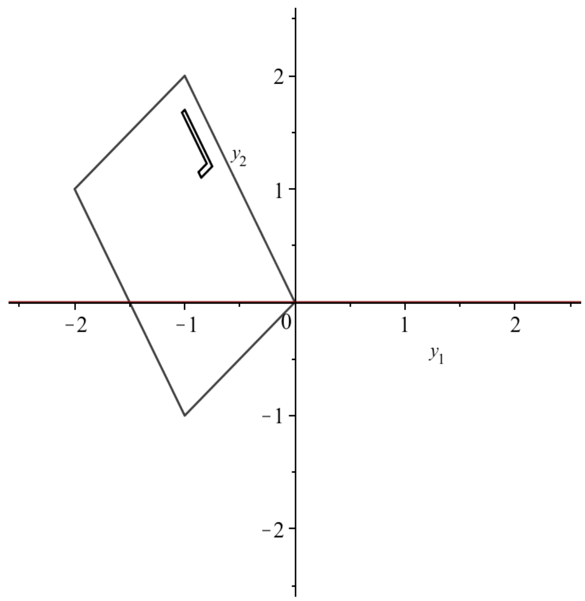
c) The linear transformation of $\mathbb{R}^3$ to itself associated to the elementary matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

is a shear, which leaves $\underline{\boldsymbol{e}}_1, \underline{\boldsymbol{e}}_2$ unchanged, but transforms $\underline{\boldsymbol{e}}_3$ to $\underline{\boldsymbol{e}}_3 + 2\underline{\boldsymbol{e}}_2$. The matrix determinant equals 1, and shears leave volume unchanged.

We showed that the determinant of a product is the product of the determinants. The volume expansion factor of a composition of matrix transformations is also the product of the individual volumn expansion factors. Since every invertible matrix is a product of elementary matrices, and since the volume expansion factors of elementary matrices equal their determinants (up to sign), it follows that the volume expansion factor for linear transformation $T(\underline{\mathbf{x}}) = A\,\underline{\mathbf{x}}$ is equal to the absolute value of the determinant of A. (This is even true when A is not invertible, since in this case the volume "expansion" factor is zero, as is the determinant.)

Example: $A = \begin{bmatrix} -1 & -1 \\ -1 & 2 \end{bmatrix}$. L-picture for $T(\underline{\mathbf{x}}) = A\,\underline{\mathbf{x}}$:



$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$
$$det(A) = -3 = 1 \cdot 3 \cdot 1 \cdot (-1)$$

The linear transformation $T(\underline{\mathbf{x}}) = A\,\underline{\mathbf{x}}$ is a composition of (1) a reflection across the x_2 axis; (2) a shear of strength -1, which preserves the x_1 axis; (3) a scaling in the x_2 direction by a factor of 3; (4) a shear of strength 1 which preserves the x_2 axis.

Topics/concepts list for exam 1

Sections 1.1-1.9, 2.1-2.3, 3.1-3.3

The matrix equation $A\,\underline{\mathbf{x}} = \underline{\boldsymbol{b}}$ arises in a number of different contexts: it can represent a system of linear equations in the unknown $\underline{\boldsymbol{x}}$; a vector linear combination equation of the columns of A, with weights given by the entries of $\underline{\boldsymbol{x}}$; in the study of the linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ given by $T(\underline{\mathbf{x}}) = A\,\underline{\mathbf{x}}$.

An essential tool is the reduced row echelon form of a matrix (augmented or unaugmented), and what it tells you about (1) the structure of solution sets to matrix equations; (2) column dependencies of a matrix. (We turned the column dependency idea around to understand why each matrix can have only one reduced row echelon form.) We also discussed matrix transformations $T(x) = A\,x$ from $\mathbb{R}^n$ to $\mathbb{R}^m$, and we focused on the geometry of linear transformations from $\mathbb{R}^2 \rightarrow \mathbb{R}^2$, where visualization helps develop intuition.

Square matrices and linear transformations from $\mathbb{R}^n \rightarrow \mathbb{R}^n$ are an important special case. Invertible matrices are useful in matrix algebra. One should know how to find matrix inverses and use them.

Determinants are useful algebraic tools and one should know how to compute them from the definition or using elementary row and column operations. They determine whether or not a matrix has an inverse, and there is a formula for the inverse matrix, based on cofactors. The absolute values of determinants are the area/volume expansion factors for the associated linear transformations.

Expect computations, true-false, example constructions, explanations.

Saying the same thing in lots of different ways (important because it ties a lot of our Chapter 1-2-3 ideas together): Can you explain why these are all equivalent? Note: many of these equivalences depend on special cases of facts/concepts that hold for $m \times n$ matrices and that you are responsible for understanding on our midterm.

<u>The invertible matrix theorem</u> (page 114, with added determinant condition)

Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

a) A is an invertible matrix.

b) The reduced row echelon form of A is the $n \times n$ identity matrix.

c) A has n pivot positions

d) The equation $A\underline{\mathbf{x}} = \underline{\mathbf{0}}$ has only the trivial solution $\underline{\mathbf{x}} = \underline{\mathbf{0}}$.

e) The columns of A form a linearly independent set.

f) The linear transformation $T(\underline{\mathbf{x}}) := A\,\underline{\mathbf{x}}$ is one-one.

g) The equation $A\,\underline{\mathbf{x}} = \underline{\boldsymbol{b}}$ has at least one solution for each $\underline{\boldsymbol{b}} \in \mathbb{R}^n$.

h) The columns of A span $\mathbb{R}^n$.

i) The linear transformation $T(\underline{\mathbf{x}}) := A\,\underline{\mathbf{x}}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.

j) There is an $n \times n$ matrix C such that $C\,A = I$.

k) There is an $n \times n$ matrix D such that $A\,\mathrm{D} = I$.

l) A^T is an invertible matrix.

m) $|A| \neq 0$