Math 2270-002 Week 5 notes

We will not necessarily finish the material from a given day's notes on that day. We may also add or subtract some material as the week progresses, but these notes represent an in-depth outline of what we plan to cover. These notes cover material in 2.1-2.3 and 3.1.

(2.4 → end of 2, some skipped, some later)

Mon Sept 17

- 2.1-2.2 Matrix algebra and matrix inverses, continued

Announcements:

'til 12:57

Warm-up Exercise: multiply these matrices, if possible

$$A_{m\times n} B_{n\times p} = A_{m\times p}$$

must agree

a) $\begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

1st col.

$$\begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2+3 \\ 3/2 - 3/2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

2nd col.

$$\begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} -4+4 \\ 3-2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

b) $\begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} -1 & 4 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 5 & 14 \end{bmatrix}$

c) $\cancel{\begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} -1 & 4 \\ 3 & 5 \end{bmatrix}} =$ DNE

On Friday we began talking about matrix algebra and matrix inverses. The matrix addition and scalar multiplication rules are just like for vectors. entry by entry

$$3\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ -3 & 14 \end{bmatrix}$$

Matrix multiplication is more interesting, and corresponds to the composition of the associated linear transformations. More precisely, we checked the following with an example and in general:

<u>Definition</u>: if B is a $p \times m$ matrix and $A = [\underline{\boldsymbol{a}}_1, \underline{\boldsymbol{a}}_2 \ldots \underline{\boldsymbol{a}}_n]$ is an $m \times n$ matrix, then $B\,A$ is a $p \times n$ matrix which can be computed column by column as

$$B\,A := [B\,\underline{\boldsymbol{a}}_1\ B\,\underline{\boldsymbol{a}}_2\ \ldots\ B\,\underline{\boldsymbol{a}}_n]\ .$$

Equivalently,

$$entry_{i\,j}\,B\,A = row_i\,(B) \cdot col_j(A).$$

<u>Theorem</u>: The matrix $B\,A$ is the matrix for the composition function $T_2 \circ T_1 : \mathbb{R}^n \to \mathbb{R}^p$, where

$$T_1 : \mathbb{R}^n \to \mathbb{R}^m,\quad T_1(\underline{\boldsymbol{x}}) = A\,\underline{\boldsymbol{x}}, \qquad (A_{m \times n}).$$

$$T_2 : \mathbb{R}^m \to \mathbb{R}^p,\quad T_2(\underline{\boldsymbol{y}}) = B\,\underline{\boldsymbol{y}}. \qquad (B_{p \times m})$$

$$\mathbb{R}^n \xrightarrow{A} \mathbb{R}^m \xrightarrow{B} \mathbb{R}^p$$

In other words,

$$T_2 \circ T_1(\underline{\boldsymbol{x}}) = B(A\,\underline{\boldsymbol{x}}) = (BA)\underline{\boldsymbol{x}}\ .$$

(example we did on Friday)

Exercise 2 Compute

$$\underset{B}{\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}}\underset{A}{\begin{bmatrix} 0 & 1 & -2 & 3 \\ 1 & 0 & 4 & 1 \end{bmatrix}} = \begin{bmatrix} 2 & 1 & 6 & 5 \\ 3 & -1 & 14 & 0 \end{bmatrix}$$

$$\mathbb{R}^4 \xrightarrow[A]{T_1} \mathbb{R}^2 \xrightarrow[B]{T_2} \mathbb{R}^2 \qquad \text{composition } T_2 \circ T_1$$

$$T_1 : \mathbb{R}^4 \to \mathbb{R}^2 \qquad T_2 : \mathbb{R}^2 \to \mathbb{R}^2$$

Exercise 3 For

$$T_1(\underline{\boldsymbol{x}}) = \begin{bmatrix} 0 & 1 & -2 & 3 \\ 1 & 0 & 4 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \qquad T_2(\underline{\boldsymbol{y}}) = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

compute $T_2\left(T_1(\underline{\boldsymbol{x}})\right)$. How does this computation relate to Exercise 2?

$$\| $$

$$\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}\left(\begin{bmatrix} 0 & 1 & -2 & 3 \\ 1 & 0 & 4 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} x_2 - 2x_3 + 3x_4 \\ x_1 + 4x_3 + x_4 \end{bmatrix}$$

$$= \begin{bmatrix} 1\cdot(x_2 - 2x_3 + 3x_4) + 2(x_1 + 4x_3 + x_4) \\ -1(x_2 - 2x_3 + 3x_4) + 3(x_1 + 4x_3 + x_4) \end{bmatrix}$$

$$= \begin{bmatrix} 2x_1 + x_2 + 6x_3 + 5x_4 \\ 3x_1 - x_2 + 14x_3 + 0x_4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 6 & 5 \\ 3 & -1 & 14 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Matrix addition and multiplication have algebra rules which are like those for scalars, except that matrix multiplication does not commute:

Check some of the following. Let I_n be the $n \times n$ identity matrix, with $I_n \underline{\mathbf{x}} = \underline{\mathbf{x}}$ for all $\underline{\mathbf{x}} \in \mathbb{R}^n$. Let A, B, C have compatible dimensions so that the indicated expressions make sense. Then

<u>a</u> $A(BC) = (AB)C$ (associative property of multiplication)

look at j^{th} columns of each side.

write $C = [\vec{c}_1\ \vec{c}_2 \cdots \vec{c}_q]$

j^{th} col of LHS $= A(\text{col}_j BC) = A(B\vec{c}_j)$

j^{th} col of RHS $= (AB)\vec{c}_j$

by composition theorem

$(A_{m\times n}\ B_{n\times p})\,C_{p\times q} = m\times q$ matrix

\# rows \# cols

$(AB)_{m\times p}$

$A_{m\times n}(BC)_{n\times q} = m\times q$ matrix too.

<u>b</u> $A(B+C) = AB + AC$ (left distributive law)

$B = [\vec{b}_1\ \vec{b}_2 \cdots \vec{b}_q]$ $C = [\vec{c}_1\ \vec{c}_2 \cdots \vec{c}_q]$, $B+C = [\vec{b}_1+\vec{c}_1\ \ \vec{b}_2+\vec{c}_2 \cdots \vec{b}_q+\vec{c}_q]$

j^{th} col. on left $= A(\vec{b}_j + \vec{c}_j)$

j^{th} col. on right $= \text{col}_j(AB) + \text{col}_j(AC) = A\vec{b}_j + A\vec{c}_j$

because matrix mult is linear

<u>c</u> $(A+B)C = AC + BC$ (right distributive law)

(you can check)

<u>d</u> $rAB = (rA)B = A(rB)$ for any scalar r.

(you can check).

<u>e</u>) If $A_{m\times n}$ then $I_m A = A$ and $A I_n = A$.

$A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

$\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

Warning: $AB \neq BA$ in general. In fact, the sizes won't even match up if you don't use square matrices.

postpone today. Cover Tuesday.

The transpose operation. One reason for considering this particular operation will be more clear by the beginning of next week, but since the text introduces it in section 2.1, we will as well.

Definition: Let $B_{m \times n} = [b_{ij}]$. Then the transpose of B, denoted by B^T is an $n \times m$ matrix defined by

$1 \le i \le m$
$1 \le j \le n$

$$entry_{ij}(B^T) := entry_{ji}(B) = b_{ji}.$$

$1 \le j \le m$
$1 \le i \le n$

The effect of this definition is to turn the columns of B into the rows of B^T:

$$entry_i(col_j(B)) = b_{ij}.$$
$$entry_i(row_j(B^T)) = entry_{ji}(B^T) = b_{ij}.$$

$col_j(B) = \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{mj} \end{bmatrix}$

And to turn the rows of B into the columns of B^T:

$$entry_j(row_i(B)) = b_{ij}$$
$$entry_j(col_i(B^T)) = entry_{ji}(B^T) = b_{ij}.$$

$row_j(B^T) = [b^T_{j1}\ b^T_{j2} \cdots b^T_{jm}]$
$= [b_{1j}\ b_{2j} \cdots b_{mj}]$

Exercise 1) explore these properties with the identity

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}.$$

$[1, 2, 3]^T = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ in some texts, makes sense

Algebra of transpose:

a $(A^T)^T = A$ ✓

b $(A + B)^T = A^T + B^T$ ✓

c for every scalar r $(rA)^T = r A^T$ ✓

$entry_{ij}(AB)^T = entry_{ji}\, AB$
$= row_j(A) \cdot col_i(B)$
$= col_j(A^T) \cdot row_i(B^T)$
$= row_i(B^T) \cdot col_j(A^T)$
$= entry_{ij}(B^T A^T)$

d (The only surprising property, so we should check it.) $(A B)^T = B^T A^T$ *

check this

if A^{-1} exists then so does $(A^T)^{-1}$. & it $= (A^{-1})^T$

one consequence: if $AA^{-1} = I$, $A^{-1}A = I$

* take transpose: $(A^{-1})^T A^T = I$, $A^T (A^{-1})^T = I$

Matrix inverses: A square matrix $A_{n \times n}$ is invertible if there is a matrix $B_{n \times n}$ so that

$$AB = BA = I.$$

In this case we call B the inverse of A, and write $B = A^{-1}$.

Remark 1: A matrix A can have at most one inverse, because if we have two candidates B, C with

$$AB = BA = I \quad \text{and also} \quad AC = CA = I$$

then

$$(BA)C = IC = C$$
$$B(AC) = BI = B$$

so since the associative property $(BA)C = B(AC)$ is true, it must be that

$$B = C.$$

Remark 2: In terms of linear transformations, if $T : \mathbb{R}^n \to \mathbb{R}^n$ is the linear transformation $T(\underline{\mathbf{x}}) = A\,\underline{\mathbf{x}}$, then saying that A has an inverse matrix is the same as saying that T has an inverse linear transformation, $T^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ with matrix B so that $T^{-1} \circ T(\underline{\mathbf{x}}) = \underline{\mathbf{x}} \quad \forall\, \underline{\mathbf{x}} \in \mathbb{R}^n$ and $T \circ T^{-1}(\underline{\mathbf{y}}) = \underline{\mathbf{y}} \quad \forall\, \underline{\mathbf{y}} \in \mathbb{R}^n$.

Exercise 2a On Friday we verified that for $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ the inverse matrix is $B = A^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$.

warmup

$$\underset{A}{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}}\underset{B}{\begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Inverse matrices can be useful in solving algebra problems. For example

Theorem: If A^{-1} exists then the only solution to $A\mathbf{x} = \mathbf{b}$ is $\mathbf{x} = A^{-1}\mathbf{b}$.

If $A\vec{x} = \vec{b}$

then $A^{-1}(A\vec{x}) = A^{-1}\vec{b}$

$(A^{-1}A)\vec{x} = A^{-1}\vec{b}$

$I\vec{x} = A^{-1}\vec{b}$

$\vec{x} = A^{-1}\vec{b}$

(and for $\vec{x} = A^{-1}\vec{b}$

$A\vec{x} = A(A^{-1}\vec{b})$

$= (AA^{-1})\vec{b}$

$= I\vec{b} = \vec{b}$ ✓

Exercise 2b On Friday we used the theorem and A^{-1} in 2a, to write down the solution to the system

$$x + 2y = 5$$
$$3x + 4y = 6$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

$\vec{x} = A^{-1}\vec{b}$.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}\begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} -4 \\ 9/2 \end{bmatrix}$$

check: $x + 2y = -4 + 9 = 5$ ✓

$3x + 4y = -12 + 18 = 6$ ✓

Corollary (of the Theorem on the previous page): If A^{-1} exists, then the reduced row echelon form of A is the identity matrix.
proof: For a square matrix, solutions to $A\,\underline{\mathbf{x}} = \underline{\boldsymbol{b}}$ always exist and are unique precisely when A reduces to the identity. When A^{-1} exists, the solutions to $A\,\underline{\mathbf{x}} = \underline{\boldsymbol{b}}$ exist and are unique. So, when A^{-1} exists, A reduces to the identity.

Exercise 3 Assuming A is a square matrix with an inverse A^{-1}, and that the matrices in the equation below have dimensions which make for meaningful equation, use matrix algebra to solve for X in terms of the other matrices:

$$XA + C = B$$

$$\Leftrightarrow \quad (XA + C) - C = B - C$$

$$XA + \underbrace{(C - C)}_{0} = B - C$$

$$XA = B - C$$

$$\Leftrightarrow \quad (XA)A^{-1} = (B - C)A^{-1}$$

$$X\underbrace{(AA^{-1})}_{I} = (B - C)A^{-1}$$

$$X = (B - C)A^{-1}$$

note in this problem must put A^{-1} on the right

But where did that formula for A^{-1} come from, in our earlier example?

<u>One Answer:</u> Consider A^{-1} as an unknown matrix, $A^{-1} = X$. We want

$$A\,X = I\,.$$

We can break this matrix equation down by the columns of $X = [\underline{\boldsymbol{x}}_1\ \underline{\boldsymbol{x}}_2]$. In the two by two case we get:

$$A\left[\begin{array}{c|c} \underline{\boldsymbol{x}}_1 & \underline{\boldsymbol{x}}_2 \end{array}\right] = \left[\begin{array}{c|c} 1 & 0 \\ 0 & 1 \end{array}\right].$$

In other words, the two columns of the inverse matrix X should satisfy

$$A\,\underline{\boldsymbol{x}}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad A\,\underline{\boldsymbol{x}}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

We can solve for both of these mystery columns at once, as we've done before when we had different right hand sides:

<u>Exercise 4:</u> Reduce the double augmented matrix

$$\left[\begin{array}{cc|c|c} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right]$$

to find the two columns of A^{-1} for the example in exercise 1.

$$\begin{array}{r|cc|cc} & 1 & 2 & 1 & 0 \\ & 3 & 4 & 0 & 1 \\ \hline & 1 & 2 & 1 & 0 \\ -3R_1 + R_2 \to R_2 & 0 & -2 & -3 & 1 \\ \hline & 1 & 2 & 1 & 0 \\ R_2/_{-2} \to R_2 & 0 & 1 & 3/2 & -1/2 \\ \hline -2R_2 + R_1 \to R_1 & 1 & 0 & -2 & 1 \\ & 0 & 1 & 3/2 & -1/2 \end{array}$$

$$X = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

For 2 × 2 matrices there's also a cool formula for inverse matrices:

<u>Theorem:</u> $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}$ exists if and only if the <u>determinant</u> D $= ad - bc$ of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is non-zero. And in this case,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

(Notice that the diagonal entries have been swapped, and minus signs have been placed in front of the off-diagonal terms. This formula should be memorized.)

<u>Exercise 5)</u> Check that the magic formula reproduces the answer you got in Exercise 4 for

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1}$$

$$\det(A) = 1\cdot 4 - 3\cdot 2 = -2$$

$$A^{-1} = \frac{1}{-2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} \checkmark$$

$$\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} ad-bc & 0 \\ 0 & \underset{da}{-bc+} \end{bmatrix} = I \right)$$

<u>Remark)</u> If $ad - bc = 0$ then A does not reduce to the identity.

Exercise 6: Will this always work? Can you find A^{-1} for

$$A := \begin{bmatrix} 1 & 5 & 1 \\ 2 & 5 & 0 \\ 2 & 7 & 1 \end{bmatrix}?$$

do the related HW problem by hand

$$AX = I$$

$$A[\vec{x}_1\ \vec{x}_2\ \vec{x}_3] = [\vec{e}_1\ \vec{e}_2\ \vec{e}_3]$$

$$\left[\begin{array}{ccc|ccc} 1 & 5 & 1 & 1 & 0 & 0 \\ 2 & 5 & 0 & 0 & 1 & 0 \\ 2 & 7 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{r} \\ -2R_1+R_2 \to R_2 \\ -2R_1+R_3 \to R_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 5 & 1 & 1 & 0 & 0 \\ 0 & -5 & -2 & -2 & 1 & 0 \\ 0 & -3 & -1 & -2 & 0 & 1 \end{array}\right]$$

$$\begin{array}{r} \\ -2R_3+R_2 \to R_2 \\ \\ \end{array} \left[\begin{array}{ccc|ccc} 1 & 5 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & -2 \\ 0 & -3 & -1 & -2 & 0 & 1 \end{array}\right]$$

$$\begin{array}{r} \\ \\ 3R_2+R_3 \to R_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 5 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & -2 \\ 0 & 0 & -1 & 4 & 3 & -5 \end{array}\right]$$

$$\begin{array}{r} \\ \\ -R_3 \to R_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 5 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & -2 \\ 0 & 0 & 1 & -4 & -3 & 5 \end{array}\right]$$

$$\begin{array}{r} -R_3+R_1 \to R_1 \\ \\ \\ \end{array} \left[\begin{array}{ccc|ccc} 1 & 5 & 0 & 5 & 3 & -5 \\ 0 & 1 & 0 & 2 & 1 & -2 \\ 0 & 0 & 1 & -4 & -3 & 5 \end{array}\right]$$

$$\begin{array}{r} -5R_2+R_1 \to R_1 \\ \\ \\ \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & -2 & 5 \\ 0 & 1 & 0 & 2 & 1 & -2 \\ 0 & 0 & 1 & -4 & -3 & 5 \end{array}\right]$$

$$\begin{bmatrix} 1 & 5 & 1 \\ 2 & 5 & 0 \\ 2 & 7 & 1 \end{bmatrix} \begin{bmatrix} -5 & -2 & 5 \\ 2 & 1 & -2 \\ -4 & -3 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$AX = I$$

$$(\quad XA = I \text{ as well})$$

Exercise 7) Will this always work? Try to find B^{-1} for $B := \begin{bmatrix} 1 & 5 & 5 \\ 2 & 5 & 0 \\ 2 & 7 & 4 \end{bmatrix}$.

Here's what happens when we try to find the three columns of B^{-1}:

$BX = I$

$$BaugI := \left[\begin{array}{ccc|ccc} 1 & 5 & 5 & 1 & 0 & 0 \\ 2 & 5 & 0 & 0 & 1 & 0 \\ 2 & 7 & 4 & 0 & 0 & 1 \end{array}\right] \qquad rref(BaugI) = \left[\begin{array}{ccc|ccc} 1 & 0 & -5 & 0 & \frac{7}{4} & -\frac{5}{4} \\ 0 & 1 & 2 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 1 & \frac{3}{4} & -\frac{5}{4} \end{array}\right]$$

inconsistent

& $rref(B) \neq I$

so B^{-1} DNE

(because when B^{-1} exists unique sol'ns to $B\vec{x} = \vec{b}$ are $\vec{x} = B^{-1}\vec{b}$ so B reduces to I)

Tues Sept 18

- 2.2-2.3 Matrix inverses

Announcements: finish A^{-1} examples in M notes

talk abt thm in today's.

§2.3 Theorem (ties together ideas in one package)

Warm-up Exercise: any HW questions?

we discussed 2.1.11

w4.1ab

and related ideas

Theorem: Let $A_{n \times n}$ be a square matrix. Then A has an inverse matrix if and only if its reduced row echelon form is the identity. In this case the algorithm illustrated in our examples from yestereday will yield A^{-1}.

explanation: By the theorem we discussed on Monday, when A^{-1}exists, the linear systems

$$A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}$$

always have unique solutions ($\underline{\boldsymbol{x}} = A^{-1}\underline{\boldsymbol{b}}$). From our previous discussions about reduced row echelon form, we know that for square matrices, solutions to such linear systems exist and are unique if and only if the reduced row echelon form of A is the identity matrix. Thus by logic, whenever A^{-1}exists, A reduces to the identity.

In this case that A does reduce to I, we search for A^{-1} as the solution matrix X to the matrix equation

$$A\,X = I$$

i.e. writing $X = \left[\underline{\boldsymbol{x}}_1\ \underline{\boldsymbol{x}}_2\ \ldots\ \underline{\boldsymbol{x}}_n\right]$ in terms of its columns, we wish to solve

$$A\left[\begin{array}{c|c|c|c} & & & \\ \underline{\boldsymbol{x}}_1 & \underline{\boldsymbol{x}}_2 & \ldots. & \underline{\boldsymbol{x}}_n \\ & & & \end{array}\right] = \left[\begin{array}{c|c|c|c} 1 & 0 & & 0 \\ 0 & 1 & & 0 \\ 0 & 0 & \ldots. & 0 \\ 0 & 0 & & 1 \end{array}\right]$$

Because A reduces to the identity matrix, we may solve for X column by column as in the examples we've worked, by using a chain of elementary row operations:

$$[\,A\,|\,I\,] \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow [I\,|\,B]\,.$$

We deduce that the columns of X are exactly the columns of B, i.e. $X = B$. Thus we know that

$$A\,B = I\,.$$

To realize that $B\,A = I$ as well, we would try to solve $B\,Y = I$ for Y, and hope $Y = A$. But we can actually verify this fact by reordering the columns of $[I\,|\,B]$ above to read $[B\,|\,I]$ and then reversing each of the elementary row operations in the first computation, i.e. create the reversed chain of elementary row operations,

$$[B\,|\,I\,] \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow [I\,|\,A\,]\,.$$

so $B\,A = I$ also holds. (This is one of those rare times when matrix multiplication actually is commuative.)

To summarize: If A^{-1} exists, then solutions $\underline{\boldsymbol{x}}$ to $A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}$ always exist and are unique, so the reduced row echelon form of A is the identity. If the reduced row echelon form of A is the identity, then A^{-1} exists, because we can find it using the algorithm above. That's exactly what the Theorem claims.

Saying the same thing in lots of different ways (important because it ties a lot of our Chapter 1-2 ideas together): Can you explain why these are all equivalent?

<u>The invertible matrix theorem</u> (page 114)

Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

a) A is an invertible matrix.

b) The reduced row echelon form of A is the $n \times n$ identity matrix.

c) A has n pivot positions

d) The equation $A\,\underline{\boldsymbol{x}} = \underline{\mathbf{0}}$ has only the trivial solution $\underline{\boldsymbol{x}} = \underline{\mathbf{0}}$.

e) The columns of A form a linearly independent set.

f) The linear transformation $T(\underline{\boldsymbol{x}}) := A\,\underline{\boldsymbol{x}}$ is one-one.

g) The equation $A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}$ has at least one solution for each $\underline{\boldsymbol{b}} \in \mathbb{R}^n$.

h) The columns of A span $\mathbb{R}^n$.

i) The linear transformation $T(\underline{\boldsymbol{x}}) := A\,\underline{\boldsymbol{x}}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.

j) There is an $n \times n$ matrix C such that $C\,A = I$.

k) There is an $n \times n$ matrix D such that A D $= I$.

l) A^T is an invertible matrix.

Wed Sept 19

- 2.2-2.3 Matrix inverses: the product of elementary matrices approach to matrix inverses

Announcements: Wed notes, after fill in the $\underline{(AB)^T = B^T A^T}$

quiz...

'til 12:57

Warm-up Exercise: Compute

$$\begin{bmatrix} 3 & 2 & -1 \end{bmatrix} \begin{bmatrix} a & b & c & d \\ A & B & C & D \\ e & f & g & h \end{bmatrix}. \quad = 1 \times 4 \text{ matrix}$$

Can you express the result as a linear combination of the rows in the 3x4 matrix?

$$= \begin{bmatrix} 3a+2A-e & 3b+2B-f & 3c+2C-g & 3d+2D-h \end{bmatrix}$$

$$= 3\begin{bmatrix} a & b & c & d \end{bmatrix} + 2\begin{bmatrix} A & B & C & D \end{bmatrix} - \begin{bmatrix} e & f & g & h \end{bmatrix}$$

Exercise 1) Show that if A, B, C are invertible matrices, then

a) $$(AB)^{-1} = B^{-1}A^{-1}.$$
b) $$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$$

a). $(AB)(B^{-1}A^{-1})$
$= A(BB^{-1})A^{-1}$
$= (AI)A^{-1}$
$= AA^{-1} = I$

b) $(AB(C)(C^{-1})B^{-1}A^{-1}) \stackrel{?}{=} I$
$A(B\cancel{I}B^{-1})A^{-1}$
$A\cancel{I}A^{-1} = I$

(all the steps)
$(AB)C = A(BC)$
so $(AB)(B^{-1}A^{-1})$
$= A(\underbrace{B(B^{-1}A^{-1})})$
$= A((BB^{-1})A^{-1})$
$= A(IA^{-1})$
$= AA^{-1}$
$= I$

Theorem The product of $n \times n$ invertible matrices is invertible, and the inverse of the product is the product of their inverses in reverse order.

Our algorithm for finding the inverse of a matrix can be reformulated in terms of a product of so-called "elementary" matrices. This product idea will pay off elsewhere. To get started, let's notice an analog of the fact that a matrix times a vector is a linear combination of the matrix columns. That was in fact how we defined matrix times vector in week 2.

Definition (from 1.4) If A is an $m \times n$ matrix, with columns $\underline{\boldsymbol{a}}_1, \underline{\boldsymbol{a}}_2, \ldots \underline{\boldsymbol{a}}_n$ (in $\mathbb{R}^m$) and if $\underline{\boldsymbol{x}} \in \mathbb{R}^n$, then $A\,\underline{\boldsymbol{x}}$ is defined to be the linear combination of the columns, with weights given by the corresponding entries of $\underline{\boldsymbol{x}}$. In other words,

$$A\,\underline{\boldsymbol{x}} = \left[\underline{\boldsymbol{a}}_1 \;\; \underline{\boldsymbol{a}}_2 \;\; \ldots \underline{\boldsymbol{a}}_n\right]\underline{\boldsymbol{x}} \coloneqq x_1\,\underline{\boldsymbol{a}}_1 + x_2\,\underline{\boldsymbol{a}}_2 + \;\ldots x_n\,\underline{\boldsymbol{a}}_n\,.$$

Theorem If we multiply a *row vector* times an $n \times m$ matrix B we get a linear combination of the *rows* of B: proof: We want to check whether

$$\underline{\boldsymbol{x}}^T B = \left[\; x_1 \;\; x_2 \;\; \cdots \;\; x_n \;\right]\begin{bmatrix} \underline{\boldsymbol{b}}_1 \\ \underline{\boldsymbol{b}}_1 \\ \vdots \\ \underline{\boldsymbol{b}}_n \end{bmatrix} = x_1\,\underline{\boldsymbol{b}}_1 + x_2\,\underline{\boldsymbol{b}}_2 + \;\ldots x_n\,\underline{\boldsymbol{b}}_n\,.$$

~~where the rows of B are given by the row vectors $\underline{\boldsymbol{b}}_1, \underline{\boldsymbol{b}}_2, \ldots \underline{\boldsymbol{b}}_n$. This proposed identity is true if and only if its transpose is a true identity. But the transpose of the left side is~~

$$\cancel{\left(\underline{\boldsymbol{x}}^T B\right)^T = B^T \left(\underline{x}^T\right)^T = B^T\,\underline{\boldsymbol{x}}}$$

$$\cancel{= \left[\underline{\boldsymbol{b}}_1^T \;\; \underline{\boldsymbol{b}}_2^T \;\; \ldots \underline{\boldsymbol{b}}_n^T\right]\underline{\boldsymbol{x}}}$$

$$\cancel{x_1\,\underline{\boldsymbol{b}}_1^T + x_2\,\underline{\boldsymbol{b}}_2^T + \;\ldots x_n\,\underline{\boldsymbol{b}}_n^T}$$

~~which is the transpose of the right side of the proposed identity. So the identity is true.~~

Q.E.D.

notationally cleaner proof of thm; illustrated in warmup:

know $A\vec{x} = [\vec{a}_1 \;\; \vec{a}_2 \cdots \vec{a}_n]\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n$

take transpose of both sides:

$$\vec{x}^T A^T = x_1\vec{a}_1^T + x_2\vec{a}_2^T + \cdots + x_n\vec{a}_n^T$$

$$[x_1 \;\; x_2 \cdots x_n]\begin{bmatrix} \vec{a}_1^T \\ \vec{a}_2^T \\ \vdots \\ \vec{a}_n^T \end{bmatrix} = x_1\vec{a}_1^T + x_2\vec{a}_2^T + \cdots + x_n\vec{a}_n^T$$

▪

Exercise 2a Use the Theorem on the previous page and work row by row on so-called "elementary matrix" E_1 on the right of the product below, to show that $E_1 A$ is the result of replacing $row_3(A)$ with $row_3(A) - 2\, row_1(A)$, and leaving the other rows unchanged:

look at what happens row by row, using previous page.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ -2a_{11}+a_{31} & -2a_{12}+a_{32} & -2a_{13}+a_{33} \end{bmatrix}$$

2b) The inverse of E_1 must undo the original elementary row operation, so must replace any $row_3(A)$ with $row_3(A) + 2\, row_1(A)$. So it must be true that

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}. \quad \overset{E_1}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \checkmark$$

Check!

2c) What 3×3 matrix E_2 can we multiply times A, in order to multiply $row_2(A)$ by 5 and leave the other rows unchanged. What is E_2^{-1} ?

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

E_2^{-1} : replace "5" with $1/5$

2d) What 3×3 matrix E_3 can we multiply time A, in order to swap $row_1(A)$ with $row_3(A)$? What is E_3^{-1} ?

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$E_3^{-1} = E_3$

Definition An *elementary matrix* E is one that is obtained by doing a single elementary row operation on the identity matrix.

Theorem Let $E_{m \times m}$ be an elementary matrix. Let $A_{m \times n}$. Then the product $E\,A$ is the result of doing the same elementary row operation to A that was used to construct E from the identity matrix.

Algorithm for finding A^{-1} re-interpreted: Suppose a sequence of elementary row operations reduces the $n \times n$ square matrix A to the identity I_n. Let the corresponding elementary matrices, in order, be given by

$$E_1, E_2, \ldots\ E_p.$$

Then we have

$$E_p\left(E_{p-1} \ldots E_2\left(E_1(A)\right) \ldots\right) = I_n$$

$$\underbrace{\left(E_p\,E_{p-1} \ldots E_2\,E_1\right)}_{\text{so, } \| \atop A^{-1}} A = I_n\,.$$

So,

$$A^{-1} = E_p\,E_{p-1} \ldots E_2\,E_1.$$

Notice that

$$E_p\,E_{p-1} \ldots E_2\,E_1 = E_p\,E_{p-1} \ldots E_2\,E_1\,I_n$$

so we have obtained A^{-1} by starting with the identity matrix, and doing the same elementary row operations to it that we did to A, in order to reduce A to the identity. I find this explanation of our original algorithm to be somewhat convoluted, but as I said, the matrix product decomposition idea is going to pay dividends elsewhere.

Also, notice that we have ended up "factoring" A into a product of elementary matrices:

$$A = \left(A^{-1}\right)^{-1} = \left(E_p\,E_{p-1} \ldots E_2\,E_1\right)^{-1} = E_1^{-1}\,E_2^{-1} \ldots E_{p-1}^{-1}\,E_p^{-1}\ .$$

Friday Sept 21

- 3.1 introduction to determinants

Announcements:

Warm-up Exercise: Look over the part of this week's HW having to do with affine transformations. (handout)

Try to construct the formula for the function which transforms "Bob" into medium-large Bob located at upper left of page for w5.2 problem

Math 2270-002
Homework due September 26.

Recall that problems which are not underlined are good for seeing if you can work with the underlying concepts; only the underlined problems need to be handed in. The Wednesday quiz will be drawn from all of these concepts and from these or related problems.

3.1 Introduction to determinants
1, **3**, **9**, 15, **25**, **27**, **29**, **31**, 32, 39, 40, 41

3.2: Properties of determinants
1, 2, 3, 4, 5, **21**, **25**, **27**, **29**, **31**, **33**, **39**

3.3: *Determinants and linear transformations; adjoint formula and Cramer's rule.*
3, 5, 13, **18**, **21**, **23**, **27**, 29, **31**

w5.1a) Use Cramer's rule to re-solve for x and y in the linear system **w4.1c** from previous homework, namely

$$\begin{bmatrix} 3 & 2 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 6 \end{bmatrix}.$$

w5.1b) Compute the determinants of the two matrices in **w4.2** from previous homework, and verify that the determinant test correctly identifies the invertible matrix. The two matrices were

$$A := \begin{bmatrix} -1 & 1 & -4 \\ -1 & -1 & 2 \\ 4 & 1 & 1 \end{bmatrix} \qquad B := \begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 3 \\ -2 & 1 & -2 \end{bmatrix}$$

w5.1c) Use the adjoint formula to re-find B^{-1} in **w5.1b**.
w5.1d) Use B^{-1} to solve the system

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 3 \\ -2 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$

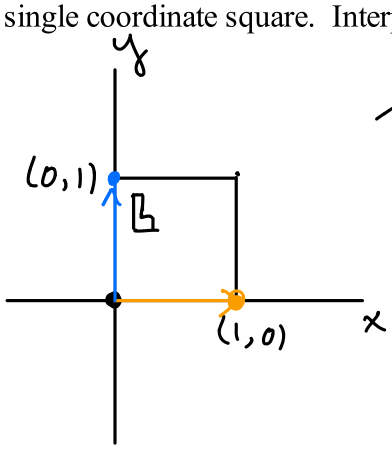
w5.1e) Re-solve for the y-variable in **w5.1d)**, using Cramer's Rule.

The following discussion and problems are related to section 1.9 and to our discussion of determinants in Chapter 3.

An *affine transformation* is a composition of a translation and a linear transformation. (When you talked about "tangent approximations" to functions in multivariable Calculus you were often talking about affine transformations. Single variable and multivariable differential calculus is built on the idea that for small scales, differentiable functions can be approximated well by affine functions.) In the following problems we'll specialize to affine transformations F from $\mathbb{R}^2$ to $\mathbb{R}^2$, i.e. functions of the form

$$F\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix}.$$

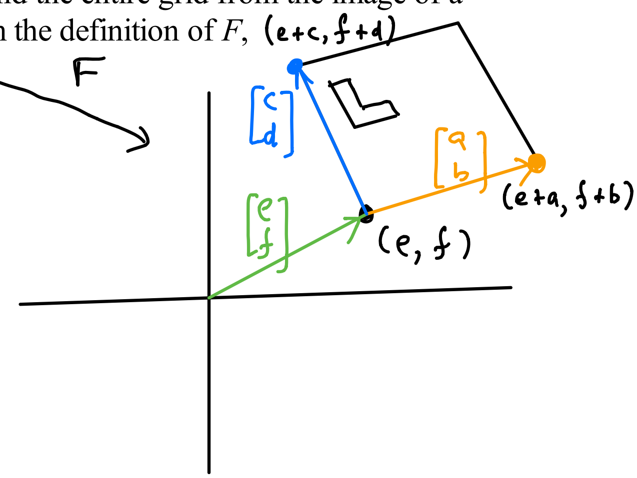
Since linear transformations transform families of parallel lines into families of parallel lines (or to points), the same is true for affine transformations. So as long as the transformations are 1-1 they will transform rectangular grid systems into parallelgram ones, and you can understand the entire grid from the image of a single coordinate square. Interpreting this algebraically, note that from the definition of F,



$$F\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} e \\ f \end{bmatrix}$$

$$F\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix}$$

$$F\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} c \\ d \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix},$$



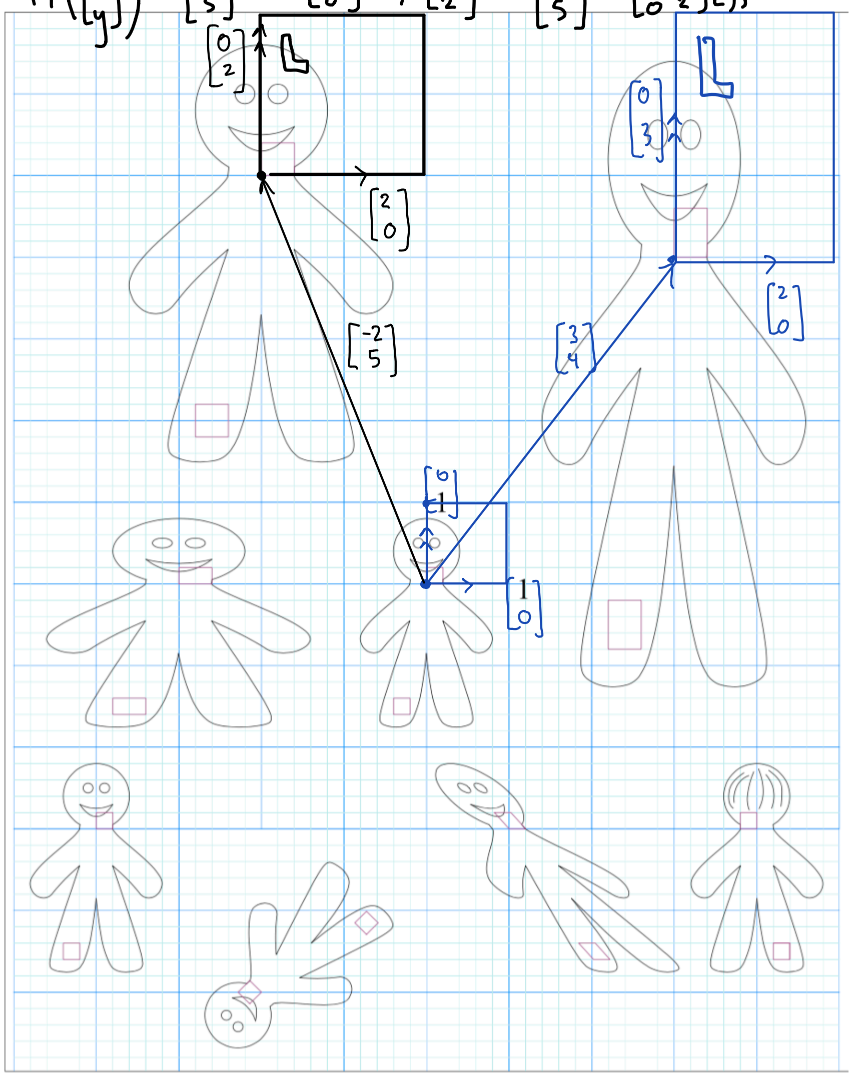
So you can reconstruct the translation vector and the two matrix columns for the affine function as soon as you know the images of $\underline{\mathbf{0}}, \underline{e}_1, \underline{e}_2$. For example, I reconstructed the transformation formula for Giant Bob in the upper right corner of the next page. Notice that Giant Bob on the next page has six times the area of original Bob - since original Bob can be filled up with different-sized squares, and the images of those squares will be rectangles having six times the original areas. There is an interesting connection between area expansion factors of affine transformations, and the determinants of the associated matrices. Recall that the determinant of a 2 by 2 matrix is given by

$$det\begin{bmatrix} a & c \\ b & d \end{bmatrix} = ad - bc.$$

Note that the determinant of Giant Bob's transformation matrix also equals 6.

warm-up

$$H\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 5 \end{bmatrix} + x\begin{bmatrix} 2 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$



$$G\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

(Think of it in this order:

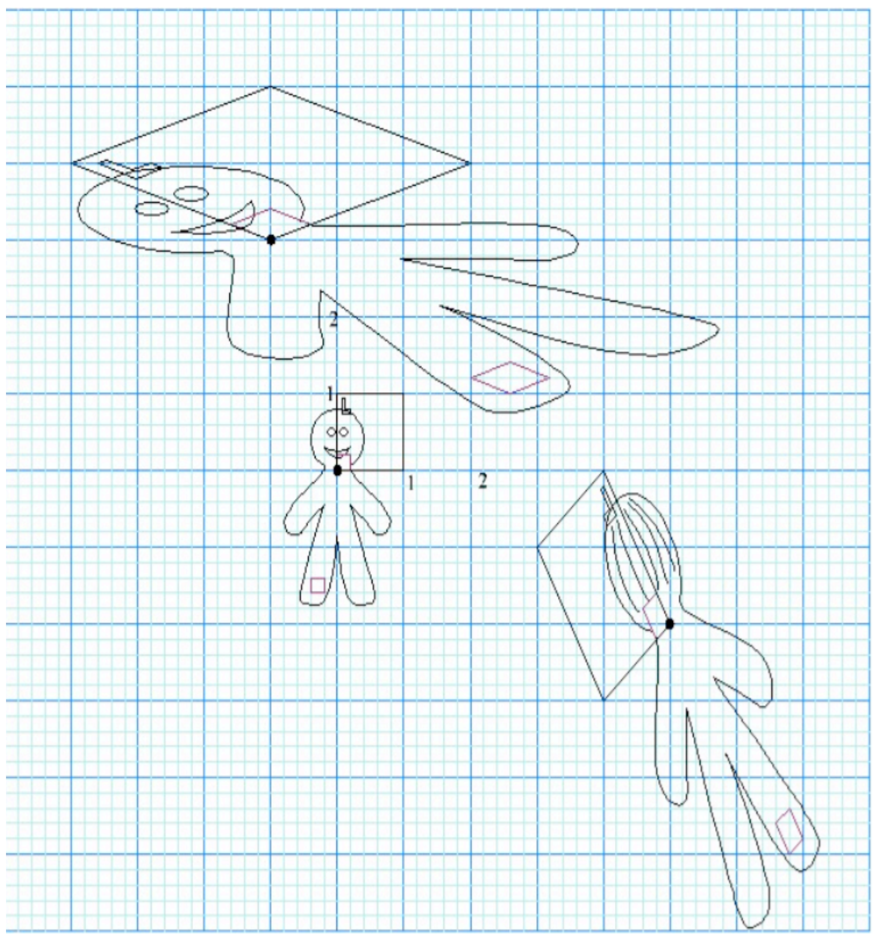
$$G\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 4 \end{bmatrix} + x\begin{bmatrix} 2 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 3 \end{bmatrix}$$)

area expansion factor $= 6$

$= \det\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

w5.2 Reconstruct the formulas for at least three more of the six (non-identity) transformations of Bob on the previous page, and comment on how the areas of the transformed Bobs are related to the determinants of the matrices in the transformations. Note that the Bob in the lower right corner got flipped over.

w5.3
a Find formulas for the two affine transformations of Bob indicated below.
b Squares in original Bob get transformed into parallelgrams in the image Bobs, and the area expansion factors are independent of the size of the original squares. So, you can deduce the area expansion factor for the image Bobs just by computing the area of the parallelgram image of the unit square. How do your area expansion factors in these two examples compare to the matrix determinants from the affine transformations?

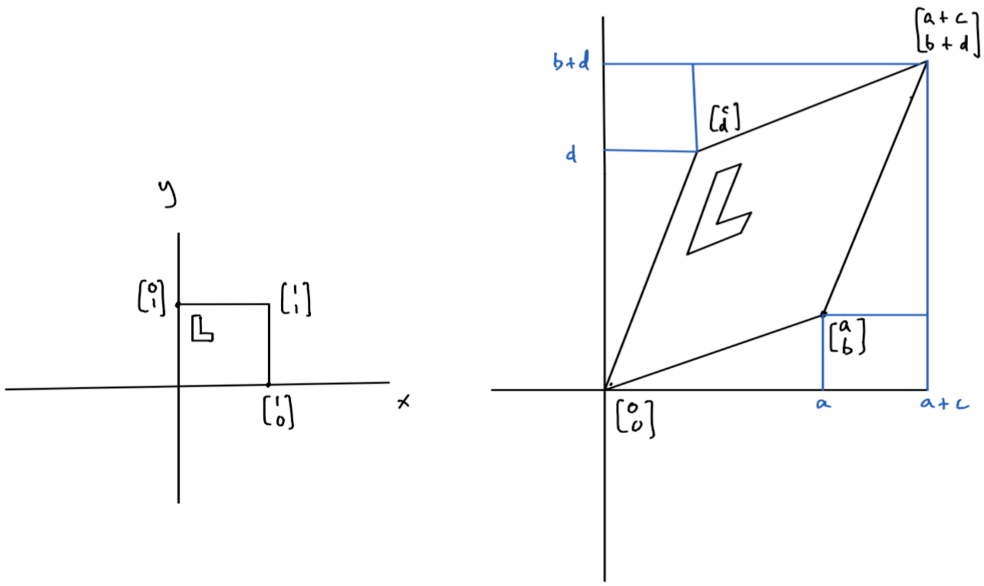


We'll talk more systematically about area/volume expansion factors and in arbitrary dimension, in class, but for affine transformations from $\mathbb{R}^2 \to \mathbb{R}^2$ one can use geometry to connect determinants to area expansion factors:

<u>**w5.4**</u> Can you compute the area of the parallelgram below (in terms of the letters a, b, c, d)? Since translations don't effect area, this will give the area expansion factor also for the images of arbitrary regions, under affine transformations that do include a translation term

$$F\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix}.$$

Hint: Start with the area of the large rectangle of length $a + c$ and height $b + d$, then subtract off the areas of the triangles and rectangles on the outside of the parallelgram. For convenience I chose the case where all of a, b, c, d are positive, and where the transformation didn't "flip" the parallelgram:



Determinants are scalars defined for square matrices $A_{n \times n}$ They always <u>determine</u> whether or not the inverse matrix A^{-1} exists, (i.e. whether the reduced row echelon form of A is the identity matrix): In fact, the determinant of A is non-zero if and only if A^{-1} exists. The determinant of a 1×1 matrix $[a_{11}]$ is defined to be the number a_{11}; determinants of 2×2 matrices are defined as in yesterday's notes; and in general determinants for $n \times n$ matrices are defined recursively, in terms of determinants of $(n-1) \times (n-1)$ submatrices:

<u>Definition:</u> Let $A_{n \times n} = [a_{ij}]$. Then the determinant of A, written $det(A)$ or $|A|$, is defined by

$$det(A) := \sum_{j=1}^{n} a_{1j}(-1)^{1+j} M_{1j} = \sum_{j=1}^{n} a_{1j} C_{1j} .$$

Here M_{1j} is the determinant of the $(n-1) \times (n-1)$ matrix obtained from A by deleting the first row and the j^{th} column, and C_{1j} is simply $(-1)^{1+j} M_{1j}$.

More generally, the determinant of the $(n-1) \times (n-1)$ matrix obtained by deleting row i and column j from A is called the <u>$i\,j$ Minor</u> M_{ij} of A, and $C_{ij} := (-1)^{i+j} M_{ij}$ is called the <u>$i\,j$ Cofactor</u> of A .

<u>Exercise 1</u> Check that the messy looking definition above gives the same answer we talked about ~~yesterday~~ in the 2×2 case, namely

earlier this week

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12} .$$

$$= a_{11}(-1)^{1+1} M_{11} + a_{12}(-1)^{1+2} M_{12}$$

$$= a_{11}(1)a_{22} + a_{12}(-1)a_{21}$$

$$= a_{11}a_{22} - a_{21}a_{12} \quad \checkmark$$

from the last page, for our convenience:

Definition: Let $A_{n \times n} = [a_{ij}]$. Then the determinant of A, written $det(A)$ or $|A|$, is defined by

$$det(A) := \sum_{j=1}^{n} a_{1j}(-1)^{1+j}M_{1j} = \sum_{j=1}^{n} a_{1j}C_{1j}.$$

Here M_{1j} is the determinant of the $(n-1) \times (n-1)$ matrix obtained from A by deleting the first row and the j^{th} column, and C_{1j} is simply $(-1)^{1+j}M_{1j}$.

Exercise 2 Work out the expanded formula for the determinant of a 3×3 matrix. It's not worth memorizing (as opposed to the recursive formula above), but it's good practice to write out at least once, and we might point to it later.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}(-1)^{1+1}M_{11} + a_{12}(-1)^{1+2}M_{12} + a_{13}(-1)^{1+3}M_{13}$$

$$= a_{11}(+1)\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12}(-1)\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{32} \end{vmatrix} + a_{13}(+1)\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{32} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

This expands to a sum of six terms. Let's organize them by whether they have + or − coefficients:

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$
$$- a_{11}a_{23}a_{32} - a_{12}a_{23}a_{31} - a_{13}a_{22}a_{31}$$

Interesting facts, which true for nxn determinants (see Wikipedia)

- each product contains exactly one entry from each row and column
- all such products are accounted for.
  (In the 3×3 case as you go down the rows there are 3 choices for the row 1 column, then two for the row 2 column, then one for the row 3 column, i.e. $3 \cdot 2 \cdot 1 = 3! = 6$ terms)
- the + or − sign depends on whether it takes an even or odd # of column interchanges to get their ordering back to (1,2,3), when the products are written as above, with rows in 1-2-3 order. (This is called the "sign" of the column permutation.)

Theorem: $det(A)$ can be computed by expanding across any row, say row i:

$$det(A) := \sum_{j=1}^{n} a_{ij}(-1)^{i+j}M_{ij} = \sum_{j=1}^{n} a_{ij}C_{ij}$$

or by expanding down any column, say column j:

$$det(A) := \sum_{i=1}^{n} a_{ij}(-1)^{i+j}M_{ij} = \sum_{i=1}^{n} a_{ij}C_{ij}.$$

Exercise 3a) Let $A := \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{bmatrix}$. Compute $det(A)$ using the definition. (On the next page we'll use other rows and columns to do the computation.)

$$|A| = a_{11}M_{11} + a_{12}(-M_{12}) + a_{13}M_{13}$$

$$= 1\begin{vmatrix} 3 & 1 \\ -2 & 1 \end{vmatrix} - 2\begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} - 1\begin{vmatrix} 0 & 3 \\ 2 & -2 \end{vmatrix}$$

$$= 1\cdot 5 - 2(-2) - 1(-6)$$

$$= 5 + 4 + 6 = 15$$

From previous page,

$$A := \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{bmatrix}.$$

<u>3b)</u> Verify that the matrix of all the cofactors of A is given by $[C_{ij}] = \begin{bmatrix} 5 & 2 & -6 \\ 0 & 3 & 6 \\ 5 & -1 & 3 \end{bmatrix}$. Then expand $det(A)$ down various columns and rows using the a_{ij} factors and C_{ij} cofactors. Verify that you always get the same value for $det(A)$, as the Theorem on the previous page guarantees. Notice that in each case you are taking the dot product of a row (or column) of A with the corresponding row (or column) of the cofactor matrix.

$$[C_{ij}] = \begin{bmatrix} +\begin{vmatrix} 3 & 1 \\ -2 & 1 \end{vmatrix} & -\begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} & +\begin{vmatrix} 0 & 3 \\ 2 & -2 \end{vmatrix} \\ -\begin{vmatrix} 2 & -1 \\ -2 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} \\ +\begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 5 & 2 & -6 \\ 0 & 3 & -3 \\ 5 & -1 & 3 \end{bmatrix}$$

$$A := \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{bmatrix} \qquad [C_{ij}] = \begin{bmatrix} 5 & 2 & -6 \\ 0 & 3 & 6 \\ 5 & -1 & 3 \end{bmatrix}$$

$row_1(A) \cdot row_1(C) =$ (previous page) $5 + 4 + 6 = 15$

$row_2(A) \cdot row_2(C) = 0 + 9 + 6 = 15$

$col_2(A) \cdot col_2(C) = 4 + 9 + 2 = 15$

3c) What happens if you take dot products between a row of A and a *different* row of $[C_{ij}]$? A column of A and a *different* column of $[C_{ij}]$? The answer may seem magic. We'll come back to this example when we talk about the magic formula for the inverses of 3×3 (or $n \times n$) invertible matrices.

$$A := \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{bmatrix} \qquad [C_{ij}] = \begin{bmatrix} 5 & 2 & -6 \\ 0 & 3 & 6 \\ 5 & -1 & 3 \end{bmatrix}$$

$row_1(A) \cdot row_2(C) = 0 + 6 - 6 = 0$

$row_3(A) \cdot row_1(C) = 10 - 4 - 6 = 0$

$col_2(A) \cdot col_1(C) = 10 + 0 - 10 = 0$

So what does AC^T equal? Note, the columns of C^T are the rows of C, so we're recomputing the various row dot products

$$AC^T = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 & 5 \\ 2 & 3 & -1 \\ -6 & 6 & 3 \end{bmatrix} = \begin{bmatrix} 15 & 0 & 0 \\ 0 & 15 & 0 \\ 0 & 0 & 15 \end{bmatrix}$$

so $A^{-1} = \frac{1}{15} C^T = \frac{1}{|A|} C^T$ where C is the cofactor matrix.

this works for $n \times n$ matrices A !!!
(we'll see why next week)

Exercise 4) Compute the following determinants by being clever about which rows or columns to use:

4a) $\begin{vmatrix} 1 & 38 & 106 & 3 \\ 0 & 2 & 92 & -72 \\ 0 & 0 & 3 & 45 \\ 0 & 0 & 0 & -2 \end{vmatrix}$; 4b) $\begin{vmatrix} 1 & 0 & 0 & 0 \\ \pi^2 & 2 & 0 & 0 \\ 0.476 & 88 & 3 & 0 \\ 1 & 22 & 33 & -2 \end{vmatrix}$.

Exercise 5) Explain why it is always true that for an upper triangular matrix (as in 2a), or for a lower triangular matrix (as in 2b), the determinant is always just the product of the diagonal entries.