Math 2270-002 Week 5 notes
We will not necessarily finish the material from a given day's notes on that day. We may also add or subtract some material as the week progresses, but these notes represent an in-depth outline of what we plan to cover. These notes cover material in 2.1-2.3 and 3.1.

Mon Sept 17
- 2.1-2.2 Matrix algebra and matrix inverses, continued

Announcements:

Warm-up Exercise:

On Friday we began talking about matrix algebra and matrix inverses. The matrix addition and scalar multiplication rules are just like for vectors.

Matrix multiplication is more interesting, and corresponds to the composition of the associated linear transformations. More precisely, we checked the following with an example and in general:

Definition: if B is a $p \times m$ matrix and $A = [\underline{\boldsymbol{a}}_1, \underline{\boldsymbol{a}}_2 \; ... \; \underline{\boldsymbol{a}}_n]$ is an $m \times n$ matrix, then $B\,A$ is a $p \times n$ matrix which can be computed column by column as

$$B\,A := [B\,\underline{\boldsymbol{a}}_1 \; B\,\underline{\boldsymbol{a}}_2 \; ... \; B\,\underline{\boldsymbol{a}}_n] .$$

Equivalently,

$$entry_{i\,j}\, B\,A = row_i\,(B) \bullet col_j(A).$$

Theorem: The matrix $B\,A$ is the matrix for the composition function $T_2 \circ T_1 : \mathbb{R}^n \rightarrow \mathbb{R}^p$, where

$$T_1 : \mathbb{R}^n \rightarrow \mathbb{R}^m, \quad T_1\,(\underline{\boldsymbol{x}}) = A\,\underline{\boldsymbol{x}}, \qquad (A_{m \times n}).$$

$$T_2 : \mathbb{R}^m \rightarrow \mathbb{R}^p, \quad T_2\,(\underline{\boldsymbol{y}}) = B\,\underline{\boldsymbol{y}}. \qquad (B_{p \times m})$$

In other words,

$$T_2 \circ T_1\,(\underline{\boldsymbol{x}}) = B(A\,\underline{\boldsymbol{x}}) = (BA)\underline{\boldsymbol{x}} .$$

Matrix addition and multiplication have algebra rules which are like those for scalars, except that matrix multiplication does not commute:

Check some of the following. Let I_n be the $n \times n$ identity matrix, with $I_n \underline{\mathbf{x}} = \underline{\mathbf{x}}$ for all $\underline{\mathbf{x}} \in \mathbb{R}^n$. Let A, B, C have compatible dimensions so that the indicated expressions make sense. Then

a $A(BC) = (AB)C$ (associative property of multiplication)

b $A(B + C) = AB + AC$ (left distributive law)

c $(A + B)C = AC + BC$ (right distributive law)

d $rAB = (rA)B = A(rB)$ for any scalar r.

e) If $A_{m \times n}$ then $I_m A = A$ and $A I_n = A$.

Warning: $AB \neq BA$ in general. In fact, the sizes won't even match up if you don't use square matrices.

The transpose operation. One reason for considering this particular operation will be more clear by the beginning of next week, but since the text introduces it in section 2.1, we will as well.

Definition: Let $B_{m \times n} = [b_{ij}]$. Then the transpose of B, denoted by B^T is an $n \times m$ matrix defined by

$$entry_{ij}(B^T) := entry_{ji}(B) = b_{ji}.$$

The effect of this definition is to turn the columns of B into the rows of B^T:

$$entry_i(col_j(B)) = b_{ij}.$$
$$entry_i(row_j(B^T) = entry_{ji}(B^T) = b_{ij}.$$

And to turn the rows of B into the columns of B^T:

$$entry_j(row_i(B)) = b_{ij}$$
$$entry_j(col_i(B^T)) = entry_{ji}(B^T) = b_{ij}.$$

Exercise 1) explore these properties with the identity

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}.$$

Algebra of transpose:

a $(A^T)^T = A$

b $(A + B)^T = A^T + B^T$

c for every scalar r $(rA)^T = r A^T$

d (The only surprising property, so we should check it.) $(A B)^T = B^T A^T$

Matrix inverses: A square matrix $A_{n \times n}$ is invertible if there is a matrix $B_{n \times n}$ so that

$$AB = BA = I.$$

In this case we call B the inverse of A, and write $B = A^{-1}$.

Remark 1: A matrix A can have at most one inverse, because if we have two candidates B, C with

$$AB = BA = I \quad \text{and also} \quad AC = CA = I$$

then

$$(BA)C = IC = C$$
$$B(AC) = BI = B$$

so since the associative property $(BA)C = B(AC)$ is true, it must be that

$$B = C.$$

Remark 2: In terms of linear transformations, if $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is the linear transformation $T(\underline{\mathbf{x}}) = A\,\underline{\mathbf{x}}$, then saying that A has an inverse matrix is the same as saying that T has an inverse linear transformation, $T^{-1} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ with matrix B so that $T^{-1} \circ T(\underline{\mathbf{x}}) = \underline{\mathbf{x}} \;\; \forall\, \underline{\mathbf{x}} \in \mathbb{R}^n$ and $T \circ T^{-1}(\underline{\mathbf{y}}) = \underline{\mathbf{y}} \;\; \forall\, \underline{\mathbf{y}} \in \mathbb{R}^n$.

Exercise 2a On Friday we verified that for $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ the inverse matrix is $B = A^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$.

$$\underset{A}{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}} \underset{B}{\begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Inverse matrices can be useful in solving algebra problems. For example

<u>Theorem:</u> If A^{-1} exists then the only solution to $A\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}$ is $\underline{\boldsymbol{x}} = A^{-1}\underline{\boldsymbol{b}}$.

<u>Exercise 2b</u> On Friday we used the theorem and A^{-1} in <u>2a</u>, to write down the solution to the system

$$x + 2y = 5$$
$$3x + 4y = 6$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix} \qquad \vec{x} = A^{-1}\vec{b}.$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}\begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} -4 \\ 9/2 \end{bmatrix}$$

check: $x + 2y = -4 + 9 = 5$ ✓
$3x + 4y = -12 + 18 = 6$ ✓

Corollary (of the Theorem on the previous page): If A^{-1} exists, then the reduced row echelon form of A is the identity matrix.
proof: For a square matrix, solutions to $A\,\underline{\mathbf{x}} = \underline{\boldsymbol{b}}$ always exist and are unique precisely when A reduces to the identity. When A^{-1} exists, the solutions to $A\,\underline{\mathbf{x}} = \underline{\boldsymbol{b}}$ exist and are unique. So, when A^{-1} exists, A reduces to the identity.

Exercise 3 Assuming A is a square matrix with an inverse A^{-1}, and that the matrices in the equation below have dimensions which make for meaningful equation, use matrix algebra to solve for X in terms of the other matrices:

$$XA + C = B$$

But where did that formula for A^{-1} come from, in our earlier example?

<u>One Answer:</u> Consider A^{-1} as an unknown matrix, $A^{-1} = X$. We want

$$A\,X = I\,.$$

We can break this matrix equation down by the columns of $X = [\underline{\boldsymbol{x}}_1\ \underline{\boldsymbol{x}}_2]$. In the two by two case we get:

$$A\left[\begin{array}{c|c} \underline{\boldsymbol{x}}_1 & \underline{\boldsymbol{x}}_2 \end{array}\right] = \left[\begin{array}{c|c} 1 & 0 \\ 0 & 1 \end{array}\right].$$

In other words, the two columns of the inverse matrix X should satisfy

$$A\,\underline{\boldsymbol{x}}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad A\,\underline{\boldsymbol{x}}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

We can solve for both of these mystery columns at once, as we've done before when we had different right hand sides:

<u>Exercise 4:</u> Reduce the double augmented matrix

$$\left[\begin{array}{cc|c|c} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right]$$

to find the two columns of A^{-1} for the example in exercise 1.

For 2×2 matrices there's also a cool formula for inverse matrices:

Theorem: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}$ exists if and only if the determinant $D = ad - bc$ of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is non-zero. And in this case,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

(Notice that the diagonal entries have been swapped, and minus signs have been placed in front of the off-diagonal terms. This formula should be memorized.)

Exercise 5) Check that the magic formula reproduces the answer you got in Exercise 4 for

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1}$$

Remark) If $ad - bc = 0$ then A does not reduce to the identity.

Exercise 6: Will this always work? Can you find A^{-1} for

$$A := \begin{bmatrix} 1 & 5 & 1 \\ 2 & 5 & 0 \\ 2 & 7 & 1 \end{bmatrix}?$$

Exercise 7) Will this always work? Try to find B^{-1} for $B := \begin{bmatrix} 1 & 5 & 5 \\ 2 & 5 & 0 \\ 2 & 7 & 4 \end{bmatrix}$.

Here's what happens when we try to find the three columns of B^{-1}:

$$BaugI := \begin{bmatrix} 1 & 5 & 5 & 1 & 0 & 0 \\ 2 & 5 & 0 & 0 & 1 & 0 \\ 2 & 7 & 4 & 0 & 0 & 1 \end{bmatrix} \qquad rref(BaugI) = \begin{bmatrix} 1 & 0 & -5 & 0 & \frac{7}{4} & -\frac{5}{4} \\ 0 & 1 & 2 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 1 & \frac{3}{4} & -\frac{5}{4} \end{bmatrix}$$

Tues Sept 18

- 2.2-2.3 Matrix inverses

Announcements:

Warm-up Exercise:

Theorem: Let $A_{n \times n}$ be a square matrix. Then A has an inverse matrix if and only if its reduced row echelon form is the identity. In this case the algorithm illustrated in our examples from yestereday will yield A^{-1}.

explanation: By the theorem we discussed on Monday, when A^{-1} exists, the linear systems

$$A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}$$

always have unique solutions ($\underline{\boldsymbol{x}} = A^{-1}\underline{\boldsymbol{b}}$). From our previous discussions about reduced row echelon form, we know that for square matrices, solutions to such linear systems exist and are unique if and only if the reduced row echelon form of A is the identity matrix. Thus by logic, whenever A^{-1} exists, A reduces to the identity.

In this case that A does reduce to I, we search for A^{-1} as the solution matrix X to the matrix equation

$$A\,X = I$$

i.e. writing $X = [\underline{\boldsymbol{x}}_1\ \underline{\boldsymbol{x}}_2\ \ldots\ \underline{\boldsymbol{x}}_n]$ in terms of its columns, we wish to solve

$$A\left[\begin{array}{c|c|c|c} \underline{\boldsymbol{x}}_1 & \underline{\boldsymbol{x}}_2 & \cdots & \underline{\boldsymbol{x}}_n \end{array}\right] = \left[\begin{array}{c|c|c|c} 1 & 0 & & 0 \\ 0 & 1 & & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & & 1 \end{array}\right]$$

Because A reduces to the identity matrix, we may solve for X column by column as in the examples we've worked, by using a chain of elementary row operations:

$$[\,A\,|\,I\,] \to \to \to \to \to [\,I\,|\,B\,].$$

We deduce that the columns of X are exactly the columns of B, i.e. $X = B$. Thus we know that

$$A\,B = I.$$

To realize that $B\,A = I$ as well, we would try to solve $B\,Y = I$ for Y, and hope $Y = A$. But we can actually verify this fact by reordering the columns of $[\,I\,|\,B\,]$ above to read $[\,B\,|\,I\,]$ and then reversing each of the elementary row operations in the first computation, i.e. create the reversed chain of elementary row operations,

$$[\,B\,|\,I\,] \to \to \to \to \to [\,I\,|\,A\,].$$

so $B\,A = I$ also holds. (This is one of those rare times when matrix multiplication actually is commuative.)

To summarize: If A^{-1} exists, then solutions $\underline{\boldsymbol{x}}$ to $A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}$ always exist and are unique, so the reduced row echelon form of A is the identity. If the reduced row echelon form of A is the identity, then A^{-1} exists, because we can find it using the algorithm above. That's exactly what the Theorem claims.

Saying the same thing in lots of different ways (important because it ties a lot of our Chapter 1-2 ideas together): Can you explain why these are all equivalent?

The invertible matrix theorem (page 114)

Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

a) A is an invertible matrix.

b) The reduced row echelon form of A is the $n \times n$ identity matrix.

c) A has n pivot positions

d) The equation $A\,\underline{\mathbf{x}} = \underline{\mathbf{0}}$ has only the trivial solution $\underline{\mathbf{x}} = \underline{\mathbf{0}}$.

e) The columns of A form a linearly independent set.

f) The linear transformation $T(\underline{\mathbf{x}}) := A\,\underline{\mathbf{x}}$ is one-one.

g) The equation $A\,\underline{\mathbf{x}} = \underline{\boldsymbol{b}}$ has at least one solution for each $\underline{\boldsymbol{b}} \in \mathbb{R}^n$.

h) The columns of A span $\mathbb{R}^n$.

i) The linear transformation $T(\underline{\mathbf{x}}) := A\,\underline{\mathbf{x}}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.

j) There is an $n \times n$ matrix C such that $C\,A = I$.

k) There is an $n \times n$ matrix D such that A D $= I$.

l) A^T is an invertible matrix.

Wed Sept 19

- 2.2-2.3 Matrix inverses: the product of elementary matrices approach to matrix inverses

Announcements:

Warm-up Exercise:

Exercise 1) Show that if A, B, C are invertible matrices, then

$$(A\,B)^{-1} = B^{-1}\,A^{-1}.$$
$$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$$

Theorem The product of $n \times n$ invertible matrices is invertible, and the inverse of the product is the product of their inverses in reverse order.

Our algorithm for finding the inverse of a matrix can be reformulated in terms of a product of so-called "elementary" matrices. This product idea will pay off elsewhere. To get started, let's notice an analog of the fact that a matrix times a vector is a linear combination of the matrix columns. That was in fact how we defined matrix times vector in week 2.

Definition (from 1.4) If A is an $m \times n$ matrix, with columns $\underline{\boldsymbol{a}}_1, \underline{\boldsymbol{a}}_2, \ldots \underline{\boldsymbol{a}}_n$ (in $\mathbb{R}^m$) and if $\underline{\boldsymbol{x}} \in \mathbb{R}^n$, then $A\,\underline{\boldsymbol{x}}$ is defined to be the linear combination of the columns, with weights given by the corresponding entries of $\underline{\boldsymbol{x}}$. In other words,

$$A\,\underline{\boldsymbol{x}} = \left[\underline{\boldsymbol{a}}_1 \;\; \underline{\boldsymbol{a}}_2 \;\; \ldots \underline{\boldsymbol{a}}_n\right]\underline{\boldsymbol{x}} \coloneqq x_1\,\underline{\boldsymbol{a}}_1 + x_2\,\underline{\boldsymbol{a}}_2 + \;\ldots x_n\,\underline{\boldsymbol{a}}_n \,.$$

Theorem If we multiply a *row vector* times an $n \times m$ matrix B we get a linear combination of the *rows* of B: proof: We want to check whether

$$\underline{\boldsymbol{x}}^T B = \left[\begin{array}{cccc} x_1 & x_2 & \cdots & x_n \end{array}\right]\left[\begin{array}{c} \underline{\boldsymbol{b}}_1 \\ \underline{\boldsymbol{b}}_1 \\ \vdots \\ \underline{\boldsymbol{b}}_n \end{array}\right] = x_1\,\underline{\boldsymbol{b}}_1 + x_2\,\underline{\boldsymbol{b}}_2 + \;\ldots x_n\,\underline{\boldsymbol{b}}_n \,.$$

where the rows of B are given by the row vectors $\underline{\boldsymbol{b}}_1, \underline{\boldsymbol{b}}_2, \ldots \underline{\boldsymbol{b}}_n$. This proposed identity is true if and only if its transpose is a true identity. But the transpose of the left side is

$$\left(\underline{\boldsymbol{x}}^T B\right)^T = B^T\left(\underline{x}^T\right)^T = B^T\,\underline{\boldsymbol{x}}$$

$$= \left[\underline{\boldsymbol{b}}_1^T \;\; \underline{\boldsymbol{b}}_2^T \;\; \ldots \underline{\boldsymbol{b}}_n^T\right]\underline{\boldsymbol{x}}$$

$$x_1\,\underline{\boldsymbol{b}}_1^T + x_2\,\underline{\boldsymbol{b}}_2^T + \;\ldots x_n\,\underline{\boldsymbol{b}}_n^T$$

which is the transpose of the right side of the proposed identity. So the identity is true.

Q.E.D.

Exercise 2a Use the Theorem on the previous page and work row by row on so-called "elementary matrix" E_1 on the right of the product below, to show that $E_1\,A$ is the result of replacing $row_3(A)$ with $row_3(A) - 2\,row_1(A)$, and leaving the other rows unchanged:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} =$$

2b) The inverse of E_1 must undo the original elementary row operation, so must replace any $row_3(A)$ with $row_3(A) + 2\,row_1(A)$. So it must be true that

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}.$$

Check!

2c) What 3×3 matrix E_2 can we multiply times A, in order to multiply $row_2(A)$ by 5 and leave the other rows unchanged. What is E_2^{-1} ?

2d) What 3×3 matrix E_3 can we multiply time A, in order to swap $row_1(A)$ with $row_3(A)$? What is E_3^{-1} ?

<u>Definition</u> An *elementary matrix* E is one that is obtained by doing a single elementary row operation on the identity matrix.

<u>Theorem</u> Let $E_{m \times m}$ be an elementary matrix. Let $A_{m \times n}$. Then the product $E\,A$ is the result of doing the same elementary row operation to A that was used to construct E from the identity matrix.

Algorithm for finding A^{-1} re-interpreted: Suppose a sequence of elementary row operations reduces the $n \times n$ square matrix A to the identity I_n. Let the corresponding elementary matrices, in order, be given by

$$E_1, E_2, \ldots\ E_p.$$

Then we have

$$E_p\left(E_{p-1} \ldots . E_2\left(E_1(A)\right) \ldots\right) = I_n$$

$$E_p\, E_{p-1} \ldots .\, E_2\, E_1\, A = I_n\,.$$

So,

$$A^{-1} = E_p\, E_{p-1} \ldots .\, E_2\, E_1.$$

Notice that

$$E_p\, E_{p-1} \ldots .\, E_2\, E_1 = E_p\, E_{p-1} \ldots .\, E_2\, E_1\, I_n$$

so we have obtained $A^{-1}$by starting with the identity matrix, and doing the same elementary row operations to it that we did to A, in order to reduce A to the identity. I find this explanation of our original algorithm to be somewhat convoluted, but as I said, the matrix product decomposition idea is going to pay dividends elsewhere.

Also, notice that we have ended up "factoring" A into a product of elementary matrices:

$$A = \left(A^{-1}\right)^{-1} = \left(E_p\, E_{p-1} \ldots .\, E_2\, E_1\right)^{-1} = E_1^{-1}\, E_2^{-1} \ldots .\, E_{p-1}^{-1}\, E_p^{-1}\ .$$

Friday Sept 21

- 3.1 introduction to determinants

Announcements:

Warm-up Exercise:

Determinants are scalars defined for square matrices $A_{n \times n}$ They always determine whether or not the inverse matrix A^{-1} exists, (i.e. whether the reduced row echelon form of A is the identity matrix): In fact, the determinant of A is non-zero if and only if A^{-1} exists. The determinant of a 1×1 matrix $\left[a_{11}\right]$ is defined to be the number a_{11}; determinants of 2×2 matrices are defined as in yesterday's notes; and in general determinants for $n \times n$ matrices are defined recursively, in terms of determinants of $(n-1) \times (n-1)$ submatrices:

Definition: Let $A_{n \times n} = \left[a_{ij}\right]$. Then the determinant of A, written $det(A)$ or $|A|$, is defined by

$$det(A) := \sum_{j=1}^{n} a_{1j}(-1)^{1+j}M_{1j} = \sum_{j=1}^{n} a_{1j}C_{1j} .$$

Here M_{1j} is the determinant of the $(n-1) \times (n-1)$ matrix obtained from A by deleting the first row and the j^{th} column, and C_{1j} is simply $(-1)^{1+j}M_{1j}$.

More generally, the determinant of the $(n-1) \times (n-1)$ matrix obtained by deleting row i and column j from A is called the ij Minor M_{ij} of A, and $C_{ij} := (-1)^{i+j}M_{ij}$ is called the ij Cofactor of A .

Exercise 1 Check that the messy looking definition above gives the same answer we talked about yesterday in the 2×2 case, namely

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}\, a_{12} .$$

from the last page, for our convenience:

Definition: Let $A_{n \times n} = [a_{ij}]$. Then the determinant of A, written $det(A)$ or $|A|$, is defined by

$$det(A) := \sum_{j=1}^{n} a_{1j}(-1)^{1+j} M_{1j} = \sum_{j=1}^{n} a_{1j} C_{1j}.$$

Here M_{1j} is the determinant of the $(n-1) \times (n-1)$ matrix obtained from A by deleting the first row and the j^{th} column, and C_{1j} is simply $(-1)^{1+j} M_{1j}$.

Exercise 2 Work out the expanded formula for the determinant of a 3×3 matrix. It's not worth memorizing (as opposed to the recursive formula above), but it's good practice to write out at least once, and we might point to it later.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} =$$

Theorem: $det(A)$ can be computed by expanding across any row, say row *i*:

$$det(A) := \sum_{j=1}^{n} a_{ij}(-1)^{i+j} M_{ij} = \sum_{j=1}^{n} a_{ij} C_{ij}$$

or by expanding down any column, say column *j*:

$$det(A) := \sum_{i=1}^{n} a_{ij}(-1)^{i+j} M_{ij} = \sum_{i=1}^{n} a_{ij} C_{ij}.$$

Exercise 3a) Let $A := \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{bmatrix}$. Compute $det(A)$ using the definition. (On the next page we'll use other rows and columns to do the computation.)

From previous page,

$$A := \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{bmatrix}.$$

3b) Verify that the matrix of all the cofactors of A is given by $\left[C_{ij}\right] = \begin{bmatrix} 5 & 2 & -6 \\ 0 & 3 & 6 \\ 5 & -1 & 3 \end{bmatrix}$. Then expand $det(A)$ down various columns and rows using the a_{ij} factors and C_{ij} cofactors. Verify that you always get the same value for $det(A)$, as the Theorem on the previous page guarantees. Notice that in each case you are taking the dot product of a row (or column) of A with the corresponding row (or column) of the cofactor matrix.

$$A := \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{bmatrix} \qquad \left[C_{ij}\right] = \begin{bmatrix} 5 & 2 & -6 \\ 0 & 3 & 6 \\ 5 & -1 & 3 \end{bmatrix}$$

3c) What happens if you take dot products between a row of A and a *different* row of $\left[C_{ij}\right]$? A column of A and a *different* column of $\left[C_{ij}\right]$? The answer may seem magic. We'll come back to this example when we talk about the magic formula for the inverses of 3×3 (or $n \times n$) invertible matrices.

$$A := \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{bmatrix} \qquad \left[C_{ij}\right] = \begin{bmatrix} 5 & 2 & -6 \\ 0 & 3 & 6 \\ 5 & -1 & 3 \end{bmatrix}$$

Exercise 4) Compute the following determinants by being clever about which rows or columns to use:

4a) $\begin{vmatrix} 1 & 38 & 106 & 3 \\ 0 & 2 & 92 & -72 \\ 0 & 0 & 3 & 45 \\ 0 & 0 & 0 & -2 \end{vmatrix}$; 4b) $\begin{vmatrix} 1 & 0 & 0 & 0 \\ \pi^2 & 2 & 0 & 0 \\ 0.476 & 88 & 3 & 0 \\ 1 & 22 & 33 & -2 \end{vmatrix}$.

Exercise 5) Explain why it is always true that for an upper triangular matrix (as in 2a), or for a lower triangular matrix (as in 2b), the determinant is always just the product of the diagonal entries.

Fri Sept 21

- 3.1 determinants

Announcements:

Warm-up Exercise:

Determinants are scalars defined for square matrices $A_{n \times n}$ They always determine whether or not the inverse matrix A^{-1} exists, (i.e. whether the reduced row echelon form of A is the identity matrix): In fact, the determinant of A is non-zero if and only if A^{-1} exists. The determinant of a 1×1 matrix $[a_{11}]$ is defined to be the number a_{11}. (And whether or not $a_{11} = 0$ determines if it doesn't or does have a multiplicative inverese.) Determinants of 2×2 matrices are defined as in or magic formula for inverse matrices, in the 2×2 case; and in general determinants for $n \times n$ matrices are defined recursively, in terms of determinants of $(n-1) \times (n-1)$ submatrices:

Definition: Let $A_{n \times n} = [a_{ij}]$. Then the determinant of A, written $det(A)$ or $|A|$, is defined by

$$det(A) := \sum_{j=1}^{n} a_{1j}(-1)^{1+j} M_{1j} = \sum_{j=1}^{n} a_{1j} C_{1j}.$$

Here M_{1j} is the determinant of the $(n-1) \times (n-1)$ matrix obtained from A by deleting the first row and the j^{th} column, and C_{1j} is simply $(-1)^{1+j} M_{1j}$.

More generally, the determinant of the $(n-1) \times (n-1)$ matrix obtained by deleting row i and column j from A is called the $i\,j$ Minor M_{ij} of A, and $C_{ij} := (-1)^{i+j} M_{ij}$ is called the $i\,j$ Cofactor of A.

Exercise 1 Check that the messy looking definition above gives the same answer we talked about in regards to our formula for the inverse of 2×2 matrices, namely

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}.$$

from the last page, for our convenience:

Definition: Let $A_{n \times n} = [a_{ij}]$. Then the determinant of A, written $det(A)$ or $|A|$, is defined by

$$det(A) := \sum_{j=1}^{n} a_{1j}(-1)^{1+j}M_{1j} = \sum_{j=1}^{n} a_{1j}C_{1j}.$$

Here M_{1j} is the determinant of the $(n-1) \times (n-1)$ matrix obtained from A by deleting the first row and the j^{th} column, and C_{1j} is simply $(-1)^{1+j}M_{1j}$.

Exercise 2 Work out the expanded formula for the determinant of a 3×3 matrix. It's not worth memorizing (as opposed to the recursive formula above), but it's good practice to write out at least once, and we might point to it later.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} =$$

Exercise 3a) Let $A := \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{bmatrix}$. Compute $det(A)$ using the definition. (On the next page we'll use other rows and columns to do the computation.)

Theorem: $det(A)$ can be computed by expanding across any row, say row i:

$$det(A) := \sum_{j=1}^{n} a_{ij}(-1)^{i+j}M_{ij} = \sum_{j=1}^{n} a_{ij}C_{ij}$$

or by expanding down any column, say column j:

$$det(A) := \sum_{i=1}^{n} a_{ij}(-1)^{i+j}M_{ij} = \sum_{i=1}^{n} a_{ij}C_{ij}.$$

(proof is not so easy - our text skips it and so will we. If you look on Wikipedia you'll see that the determinant is actually a sum of n factorial terms, each of which is $\pm$ a product of n entries of A where each product has exactly one entry from each row and column. The $\pm$ sign has to do with whether the corresponding permutation is even or odd. You can verify this pretty easily for the 2×2 and 3×3 cases. Then one shows inductively that each row or column cofactor expansion reproduces this sum, in the $n \times n$ case.)

From previous page,

$$A := \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{bmatrix}.$$

3b) Verify that the matrix of all the cofactors of A is given by $\left[C_{i\,j}\right] = \begin{bmatrix} 5 & 2 & -6 \\ 0 & 3 & 6 \\ 5 & -1 & 3 \end{bmatrix}$. Then expand $det(A)$ down various columns and rows using the $a_{i\,j}$ factors and $C_{i\,j}$ cofactors. Verify that you always get the same value for $det(A)$, as the Theorem on the previous page guarantees. Notice that in each case you are taking the dot product of a row (or column) of A with the corresponding row (or column) of the cofactor matrix.

$$A := \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{bmatrix} \qquad \left[C_{i\,j}\right] = \begin{bmatrix} 5 & 2 & -6 \\ 0 & 3 & 6 \\ 5 & -1 & 3 \end{bmatrix}$$

3c) What happens if you take dot products between a row of A and a *different* row of $[C_{i,j}]$? A column of A and a *different* column of $[C_{i,j}]$? The answer may seem magic.

$$A := \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{bmatrix} \qquad [C_{i,j}] = \begin{bmatrix} 5 & 2 & -6 \\ 0 & 3 & 6 \\ 5 & -1 & 3 \end{bmatrix}$$

3d) The adjoint matrix is defined to be the transpose of the cofactor matrix. So in our example,

$$A := \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{bmatrix} \qquad adj(A) = (cof(A))^T = \begin{bmatrix} 5 & 0 & 5 \\ 2 & 3 & -1 \\ -6 & 6 & 3 \end{bmatrix}.$$

Reinterpret your work in 3bc to say that

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 & 5 \\ 2 & 3 & -1 \\ -6 & 6 & 3 \end{bmatrix} = \begin{bmatrix} 15 & 0 & 0 \\ 0 & 15 & 0 \\ 0 & 0 & 15 \end{bmatrix}.$$

So, in this case - and in fact always, the magic formula for A^{-1} is given by

$$A^{-1} = \frac{1}{det(A)} adj(A).$$

It seems like magic now, but we'll be able to understand why it's true after we learn about more determinant properties on Wednesday and Friday.

Exercise 4) Compute the following determinants by being clever about which rows or columns to use:

4a) $\begin{vmatrix} 1 & 38 & 106 & 3 \\ 0 & 2 & 92 & -72 \\ 0 & 0 & 3 & 45 \\ 0 & 0 & 0 & -2 \end{vmatrix}$; 4b) $\begin{vmatrix} 1 & 0 & 0 & 0 \\ \pi^2 & 2 & 0 & 0 \\ 0.476 & 88 & 3 & 0 \\ 1 & 22 & 33 & -2 \end{vmatrix}$.

Exercise 5) Explain why it is always true that for an upper triangular matrix (as in 2a), or for a lower triangular matrix (as in 2b), the determinant is always just the product of the diagonal entries.