Math 2270-002 Week 3 notes
We will not necessarily finish the material from a given day's notes on that day. We may also add or subtract some material as the week progresses, but these notes represent an outline of what we plan to cover. These notes cover material in 1.6-1.8.

Tues Sept 4
- $\bigcirc$1.6 $\bigcirc$1.7 applications; introduction to linear dependence and independence.

Announcements: part of next week's HW:

- §1.7 1, ⑤, ⑦, ⑮, ⑰, ㉑ (more later)
- HW due tomorrow (office hours after class today LCB 225) (MTW)
- Quiz tomorrow

Warm-up Exercise: Preview applications exercises 1 & 2, taken from text §1.6, in today's notes network problems → linear algebra

Exercise 1) Consider the following traffic flow problem (from our text): What are the possible flow patterns, based on the given information and that the streets are one-way, so none of the flow numbers can be negative?

EXAMPLE 2 The network in Figure 2 shows the traffic flow (in vehicles per hour) over several one-way streets in downtown Baltimore during a typical early afternoon. Determine the general flow pattern for the network.

flow in = flow out

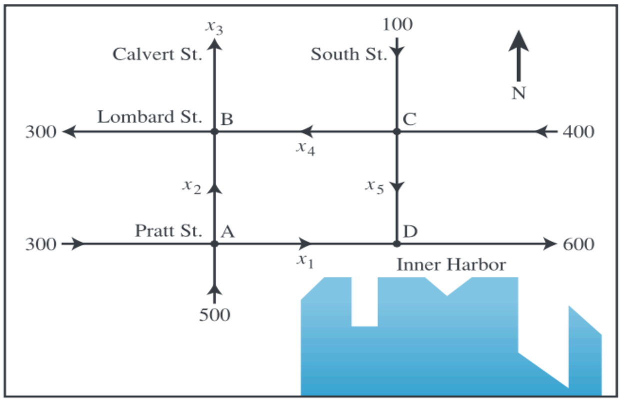

FIGURE 2 Baltimore streets.

A $800 = x_1 + x_2$

B $x_2 + x_4 = 300 + x_3$

C $500 = x_4 + x_5$

D $x_1 + x_5 = 600$

$$\begin{aligned} x_1 + x_2 \qquad\qquad &= 800 \\ x_2 - x_3 + x_4 \quad &= 300 \\ x_4 + x_5 &= 500 \\ x_1 \qquad\qquad + x_5 &= 600 \end{aligned}$$

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 800 \\ 0 & 1 & -1 & 1 & 0 & 300 \\ 0 & 0 & 0 & 1 & 1 & 500 \\ 1 & 0 & 0 & 0 & 1 & 600 \end{array}\right]$$

Hint: If you set up the flow equations for intersections A, B, C, D in that order, the following reduced row echelon form computation may be helpful:

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 800 \\ 0 & 1 & -1 & 1 & 0 & 300 \\ 0 & 0 & 0 & 1 & 1 & 500 \\ 1 & 0 & 0 & 0 & 1 & 600 \end{array}\right] \rightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 1 & 600 \\ 0 & 1 & 0 & 0 & -1 & 200 \\ 0 & 0 & 1 & 0 & 0 & 400 \\ 0 & 0 & 0 & 1 & 1 & 500 \end{array}\right]$$

$x_1 = 600 - x_5$

$x_2 = 200 + x_5$

$x_3 = 400$

$x_4 = 500 - x_5$

x_5 = free

Note: $0 \le x_5 \le 500$ since all flow rates ≥ 0

Exercise 2)

A Homogeneous System in Economics

WEB The system of 500 equations in 500 variables, mentioned in this chapter's introduction, is now known as a Leontief "input–output" (or "production") model.[1] Section 2.6 will examine this model in more detail, when more theory and better notation are available. For now, we look at a simpler "exchange model," also due to Leontief.

Suppose a nation's economy is divided into many sectors, such as various manufacturing, communication, entertainment, and service industries. Suppose that for each sector we know its total output for one year and we know exactly how this output is divided or "exchanged" among the other sectors of the economy. Let the total dollar value of a sector's output be called the **price** of that output. Leontief proved the following result.

> There exist *equilibrium prices* that can be assigned to the total outputs of the various sectors in such a way that the income of each sector exactly balances its expenses.

The following example shows how to find the equilibrium prices.

EXAMPLE 1 Suppose an economy consists of the Coal, Electric (power), and Steel sectors, and the output of each sector is distributed among the various sectors as shown in Table 1, where the entries in a column represent the fractional parts of a sector's total output.

The second column of Table 1, for instance, says that the total output of the Electric sector is divided as follows: 40% to Coal, 50% to Steel, and the remaining 10% to Electric. (Electric treats this 10% as an expense it incurs in order to operate its

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[1] See Wassily W. Leontief, "Input–Output Economics," *Scientific American*, October 1951, pp. 15–21.

business.) Since all output must be taken into account, the decimal fractions in each column must sum to 1.

Denote the prices (i.e., dollar values) of the total annual outputs of the Coal, Electric, and Steel sectors by p_C, p_E, and p_S, respectively. If possible, find equilibrium prices that make each sector's income match its expenditures.

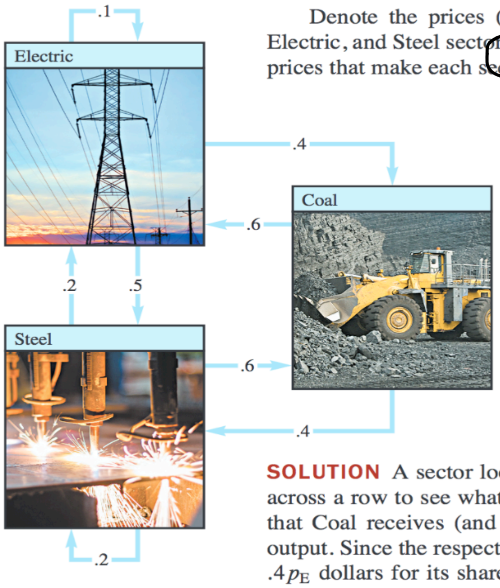


TABLE 1 A Simple Economy

Distribution of Output from:			
Coal	Electric	Steel	Purchased by:
.0	.4	.6	Coal
.6	.1	.2	Electric
.4	.5	.2	Steel

SOLUTION A sector looks down a column to see where its output goes, and it looks across a row to see what it needs as inputs. For instance, the first row of Table 1 says that Coal receives (and pays for) 40% of the Electric output and 60% of the Steel output. Since the respective values of the total outputs are p_E and p_S, Coal must spend $.4p_E$ dollars for its share of Electric's output and $.6p_S$ for its share of Steel's output. Thus Coal's total expenses are $.4p_E + .6p_S$. To make Coal's income, p_C, equal to its expenses, we want

$$p_C = .4p_E + .6p_S \tag{1}$$

$P_c = .4P_E + .6P_s$ $\quad -P_c + .4P_E + .6P_s = 0$

$P_E = .6P_c + .1P_E + .2P_s$

$P_s = .4P_c + .5P_E + .2P_s$

$$\left[\begin{array}{ccc|c} -1. & .4 & .6 & 0 \\ .6 & -.9 & .2 & 0 \\ .4 & .5 & -.8 & 0 \end{array}\right]$$

Hint: After we set up the problem, the following computation will help with the answer:

$$\left[\begin{array}{ccc|c} -1. & .4 & .6 & 0 \\ .6 & -.9 & .2 & 0 \\ .4 & .5 & -.8 & 0 \end{array}\right] \quad \text{approximately reduces to} \quad \begin{array}{cccc} p_c & p_E & p_s & \\ \hline 1. & -0. & -0.94 & 0 \\ 0. & 1. & -0.85 & 0 \\ 0. & 0. & 0. & 0 \end{array}$$

$p_C = .94 p_s$
$p_E = .85 p_s$
$p_s =$ free

$$\begin{bmatrix} p_C \\ p_E \\ p_s \end{bmatrix} = p_s \begin{bmatrix} .94 \\ .85 \\ 1 \end{bmatrix}$$

§1.6 just gives a tiny taste of applications linear independence & dependence for sets of vectors.

1.7 When we are discussing the span of a collection of vectors $\underline{v}_1, \underline{v}_2, \ldots \underline{v}_n$ we would like to know that we are being efficient in describing this span, and not wasting any free parameters because of redundancies in the vectors. For example, the most efficient way to describe a plane in $\mathbb{R}^3$ is as the span of exactly two vectors, rather than as the span of three or more. This has to do with the concept of "linear independence":

Definition:

logical opposite of linearly dependent

a) An indexed set of vectors $\{\underline{v}_1, \underline{v}_2, \ldots \underline{v}_n\}$ is said to be linearly independent if ② no one of the vectors is a linear combination of (some) of the other vectors. The concise way to say this is that the only way $\underline{\mathbf{0}}$ can be expressed as a linear combination of these vectors,

$$c_1\underline{v}_1 + c_2\underline{v}_2 + \ldots + c_n\underline{v}_n = \underline{\mathbf{0}},$$

① opposite of ① for dependent

is for all of the weights $c_1 = c_2 = \ldots = c_n = 0$.

logical negation of ② for dependent

start here.

b) An indexed set of vectors $\{\underline{v}_1, \underline{v}_2, \ldots \underline{v}_n\}$ is said to be linearly dependent if at least one of these vectors *is* a linear combination of (some) of the other vectors. The concise way to say this is that there *is* some way to write $\underline{\mathbf{0}}$ as a linear combination of these vectors

② one $\vec{v}_j$ is a linear combo of (some) of the rest

$$① \quad c_1\underline{v}_1 + c_2\underline{v}_2 + \ldots + c_n\underline{v}_n = \underline{\mathbf{0}}$$

where *not all* of the $c_j = 0$. (We call such an equation a linear dependency. Note that if we have any such linear dependency, then any $\underline{v}_j$ with $c_j \neq 0$ is a linear combination of the remaining $\underline{v}_k$ with $k \neq j$. We say that such a $\underline{v}_j$ is linearly dependent on the remaining $\underline{v}_k$.)

If ① is true, $c_j \neq 0$. then solve for $\vec{v}_j$ in ②

$$c_j\vec{v}_j = \underbrace{-c_1\vec{v}_1 - c_2\vec{v}_2 - \ldots - c_n\vec{v}_n}_{\text{no } \vec{v}_j \text{ terms.}}$$

so $\vec{v}_j = -\frac{c_1}{c_j}\vec{v}_1 - \frac{c_2}{c_j}\vec{v}_2 - \ldots - \frac{c_n}{c_j}\vec{v}_n$.

If ② is true, i.e. $\vec{v}_j = \underbrace{d_1\vec{v}_1 + d_2\vec{v}_2 + \ldots + d_n\vec{v}_n}_{\text{no } \vec{v}_j\text{'s.}}$

then $\vec{0} = d_1\vec{v}_1 + d_2\vec{v}_2 + \ldots + d_n\vec{v}_n - \vec{v}_j$ i.e. ① is true.

<u>Note:</u> The only set of a single vector $\{\underline{\boldsymbol{v}}_1\}$ that is dependent is if $\underline{\boldsymbol{v}}_1 = \underline{\mathbf{0}}$. The only sets of two non-zero vectors, $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2\}$ that are linearly dependent are when one of the vectors is a scalar multiple of the other one. For more than two vectors the situation is more complicated.

if dep. $\Rightarrow c_1\vec{v}_1 = \vec{0} \quad c_1 \neq 0$

$\Rightarrow \vec{v}_1 = \vec{0}$.

& for the zero vector $\vec{0}$

$1\cdot\vec{0} = \vec{0}$

is a dependency.

If $\{\vec{v}_1, \vec{v}_2\}$ linearly dependent

either $\vec{v}_2 = c_1\vec{v}_1$

(or $\vec{v}_1 = c_2\vec{v}_2$)

So one of them is a multple of the other.

<u>Exercise 3a)</u> Is this set linearly dependent or independent? $\left\{\begin{bmatrix}0\\0\end{bmatrix}\right\}$? How does your answer generalize to any set of vectors which includes the zero vector?

$\uparrow$

dependent

$1\begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

e.g. $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}\right\}$

$$0\begin{bmatrix}1\\0\end{bmatrix} + 0\begin{bmatrix}0\\1\end{bmatrix} + 1\cdot\begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

dependent.

<u>3b</u>) Is this set of vectors linearly dependent or independent? $\left\{\underset{\vec{v}_1}{\begin{bmatrix}-2\\1\end{bmatrix}}, \underset{\vec{v}_2}{\begin{bmatrix}6\\-3\end{bmatrix}}\right\}$?

$$\left.\begin{array}{rl} & -3\vec{v}_1 = \vec{v}_2 \\ & 3\vec{v}_1 + \vec{v}_2 = \vec{0} \\ \text{or} & -3\vec{v}_1 - \vec{v}_2 = \vec{0} \\ & 6\vec{v}_1 + 2\vec{v}_2 = \vec{0} \end{array}\right\} \text{any one suffices}$$

here after def of dependence

Example

The set of vectors $\left\{ \underline{\boldsymbol{v}}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \underline{\boldsymbol{v}}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \underline{\boldsymbol{v}}_3 = \begin{bmatrix} -2 \\ 8 \end{bmatrix} \right\}$ in $\mathbb{R}^2$ is linearly dependent because, as we showed when we were introducing vector equations (and as we can quickly recheck),

$$-3.5 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + 1.5 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 8 \end{bmatrix}.$$

$\vec{v}_3 = -3.5\,\vec{v}_1 + 1.5\,\vec{v}_2$ ②

We can also write this linear dependency as

$$-3.5\underline{\boldsymbol{v}}_1 + 1.5\underline{\boldsymbol{v}}_2 - \underline{\boldsymbol{v}}_3 = \underline{\boldsymbol{0}}$$

① dependent

(or any non-zero multiple of that equation.)

two ways of showing dependent

but $\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$ is independent

Remark: If we're studying the span of a set of vectors, we'd prefer to be dealing with independent ones in order to avoid redundancies in how we represent a given vector as a linear combination. If you look at the example above, we can delete the vector v_3 (or any one of the other two vectors in this example), without shrinking the span:

$$span\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \underline{\boldsymbol{v}}_3\} = span\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2\}.$$

The reason for this is that

$$c_1\underline{\boldsymbol{v}}_1 + c_2\underline{\boldsymbol{v}}_2 + c_3\underline{\boldsymbol{v}}_3 = c_1\underline{\boldsymbol{v}}_1 + c_2\underline{\boldsymbol{v}}_2 + c_3\left(-3.5\,\boldsymbol{v}_1 + 1.5\,\underline{\boldsymbol{v}}_2\right) = d_1\underline{\boldsymbol{v}}_1 + d_2\underline{\boldsymbol{v}}_2\,.$$

Wed Sept 5

- 1.7 Linear dependence/independence continued.

Announcements:

- quiz today
- usually you'll get back all assignments within a week (or sooner). HW delayed this week
- HW3 is mostly posted. All by tonight

Warm-up Exercise: Exercise 1 in today's notes

$c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n = \vec{0}$ has non-zero solutions $\vec{c}$

Note that the set of vectors $\{\underline{v}_1, \underline{v}_2, \ \dots \ \underline{v}_n\}$ is linearly dependent if and only if there are non-zero solutions $\underline{c}$ to the homogeneous matrix equation

$$A\,\underline{c} = \underline{0}$$

for the matrix $A = [\underline{v}_1 \ \underline{v}_2 \ \dots \ \underline{v}_n]$ having the given vectors as columns. Thus all linear independence/dependence questions can be answered using reduced row echelon form and facts about homogeneous matrix solutions.

Warmup problem

Exercise 1) Show that the vectors

$$\underline{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \ \underline{v}_2 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \ \underline{v}_3 = \begin{bmatrix} -1 \\ 6 \\ 4 \end{bmatrix}$$

are linearly dependent (even though no two of them are scalar multiples of each other). What does this mean geometrically about the span of these three vectors?

to solve

$$c_1\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + c_2\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + c_3\begin{bmatrix} -1 \\ 6 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

are there sol'ns besides $c_1 = c_2 = c_3 = 0$?

Answer: dependent

Hint: You might find this computation useful:

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 2 & 6 & 0 \\ 2 & 0 & 4 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

OR!

$c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{v}_3$?

$$\left[\begin{array}{cc|c} 1 & -1 & -1 \\ 0 & 2 & 6 \\ 2 & 0 & 4 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right]$$

So $2\vec{v}_1 + 3\vec{v}_2 = \vec{v}_3$

there are non-trivial solutions

- c_3 is a free variable (∞'ly many sol'ns)

more specific:

$c_1 = -2c_3$
$c_2 = -3c_3$
$c_3 = $ free

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = c_3\begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}.$$

e.g. $-2\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - 3\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 6 \\ 4 \end{bmatrix} = \vec{0}$

<u>Exercise 2)</u> Are the vectors

$$\underline{\boldsymbol{v}}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \ \underline{\boldsymbol{v}}_2 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \ \underline{\boldsymbol{w}}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

linearly independent or dependent? Hint:

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ 2 & 0 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right].$$

$$\uparrow \qquad\qquad\qquad\qquad \Downarrow$$

vector eqtn

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0} \Longrightarrow \begin{array}{l} c_1 = 0 \\ c_2 = 0 \\ c_3 = 0 \end{array}$$

Independent

Exercise 3a) Why must more than three vectors in ℝ³ be linearly dependent?

consider $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + c_4\vec{v}_4 + \cdots + c_n\vec{v}_n = \vec{0}$ $n \geq 4$ vectors

augmented matrix
$$3\left\{\underbrace{\begin{bmatrix}\vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n\end{bmatrix}}_{n}\begin{matrix}0\\0\\0\end{matrix}\right. \longrightarrow \underbrace{\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}}\begin{matrix}0\\0\\0\end{matrix}$$

at most 3 pivots
at least $n-3$ free variables

so, lots of dependencies.

3b) How about more than *m* vectors in ℝ*ᵐ* ?

$$m\left\{\underbrace{\begin{bmatrix}\vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n\\ & & & \\ & & & \end{bmatrix}}_{n>m}\right.$$

at most m pivots
at least $n-m$ free variables

so, lots of dependencies

3c) If you are given a set of exactly *n* vectors in ℝⁿ how can you check whether or not they are linearly independent?

$$n\left\{\underbrace{\left[\begin{array}{c|c|c|c}\vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n\\ & & & \\ & & & \end{array}\right]}_{n}\right. \quad \text{rref} \begin{cases}\begin{bmatrix}1 & 0 & 0 & - \\ 0 & 1 & 0 & - \\ \vdots & & & \\ 0 & \cdots & 0 & 1\end{bmatrix} \text{ “}I\text{”} & \underline{\text{vectors independent}} \\ < n \text{ pivots} & \end{cases}$$

so non-pivot cols, so free variables

so dependent

$c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n = \vec{0}$

3d) If you have a set of fewer than *n* vectors in ℝⁿ can they span ℝⁿ?

Friday warm-up

$p<n$: $x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_p\vec{v}_p = \vec{b} \in \mathbb{R}^n$ NO

$$n\left\{\left[\underbrace{\begin{matrix}\vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_p\end{matrix}}_{p<n}\ \vdots\ \vec{b}\right]\right. \longrightarrow \left[\begin{array}{c|c} & c_1\\ & c_2\\ & \vdots\\ 0\ 0\ 0 \cdots 0 & c_n\end{array}\right]$$

inconsistent unless $c_n = 0$.

at most p pivots in coeff matrix $p<n$, so bottom row of coeff matrix reduces to 0

3e) If $\{\underline{v}_1, \underline{v}_2, \ldots \underline{v}_n\}$ is a set of exactly *n* vectors in ℝⁿ what condition on the reduced row echelon form of $A = [\underline{v}_1, \underline{v}_2, \ldots \underline{v}_n]$ guarantees and is required so that the vectors span ℝⁿ ? Compare with 3c.

Friday warm-up

if exactly n vectors in $\mathbb{R}^n$

need rref $A = I$ in order for the vectors to span $\mathbb{R}^n$

(else there's a bottom row of 0's in rref(A).

Fri Sept 7

- 1.7 reduced row echelon form as encoding linear independence/dependence of matrix columns; introduction to linear transformations, section 1.8.

Announcements: returned HW 1 & Quiz 3

↑

only some of the problems are graded
– see rubric on public HW page
solutions to non-text problems are on CANVAS.

'til 12:57

Warm-up Exercise: 3d, 3e; the last two exercises in yesterday's notes

Exercise 1) Consider the homogeneous matrix equation $A\,\underline{x} = \underline{0}$, with the matrix A (and its reduced row echelon form) shown below:

$A_{3\times 5}$ variables $x_1, x_2, \ldots x_5$

$$A := \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 1 & 1 & 0 \\ 2 & 4 & 1 & 4 & 1 & 0 \\ -2 & -4 & 0 & -2 & -2 & 0 \\ \vec{a}_1 & \vec{a}_2 & \vec{a}_3 & \vec{a}_4 & \vec{a}_5 & \end{array}\right] \rightarrow \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

in class I accidentally used "$\vec{a}$" for the columns, which disagreed with how I named them here ←

Find and express the solutions to this system in linear combination form. Note that you are finding all of the dependencies for the collection of vectors that are the columns of A, namely the set

$$\left\{ \overset{\vec{a}_1\,\|}{\underline{v}_1} = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}, \overset{\vec{a}_2\,\|}{\underline{v}_2} = \begin{bmatrix} 2 \\ 4 \\ -4 \end{bmatrix}, \overset{\vec{a}_3\,\|}{\underline{v}_3} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \overset{\vec{a}_4\,\|}{\underline{v}_4} = \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}, \overset{\vec{a}_5\,\|}{\underline{v}_5} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \right\}.$$

$x_1 = -2x_2 - x_4 - x_5$
$x_2 = x_2$ free
$x_3 = -2x_4 + x_5$
$x_4 = x_4$ free
$x_5 = x_5$ free

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

these homog. sol'tns also encode column dependencies

e.g. $x_2 = 1,\ x_4 = x_5 = 0 \Rightarrow \boxed{-2\vec{a}_1 + 1\,\vec{a}_2 = \vec{0}}$

$$-2\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} + 1\cdot\begin{bmatrix} 2 \\ 4 \\ -4 \end{bmatrix} = \vec{0} \ ✓$$

$\vec{v}_2 = 2\vec{v}_1$ next page

e.g. $x_2 = x_5 = 0,\ x_4 = 1 \Longrightarrow -\vec{a}_1 - 2\vec{a}_3 + \vec{a}_4 = \vec{0}$

e.g. $x_2 = x_4 = 0,\ x_5 = 1 \Rightarrow -\vec{a}_1 + \vec{a}_3 + \vec{a}_5 = \vec{0}$

$\vec{v}_4 = \vec{v}_1 + 2\vec{v}_3$ next page

$\vec{v}_5 = \vec{v}_1 - \vec{v}_3$ next page

Exercise 2) Focus on the idea that solutions to homogeneous matrix equations correspond exactly to linear dependencies between the columns of the matrix. Now, since the solution set to a homogeneous linear system does not change as you do elementary row operations to the augmented matrix, column dependencies also do not change. Therefore the vectors that span the space of homogeneous solutions in Exercise 1 are encoding the key column dependencies in $\mathbb{R}^3$, for both the original matrix, and for the reduced row echelon form.

Check this, by reading off "easy" column dependencies in the reduced matrix; seeing that they are also dependencies in the original matrix; and that they correspond to the basis of the homogeneous solution space. Magic! We will use this magic in important interesting ways, later in the course.

$$A := \begin{array}{c} \left[\begin{array}{ccccc} 1 & 2 & 0 & 1 & 1 \\ 2 & 4 & 1 & 4 & 1 \\ -2 & -4 & 0 & -2 & -2 \end{array}\right] \\ \begin{array}{ccccc} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 & \vec{v}_5 \end{array} \end{array} \rightarrow \begin{array}{c} \left[\begin{array}{ccccc} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{ccccc} \vec{w}_1 & \vec{w}_2 & \vec{w}_3 & \vec{w}_4 & \vec{w}_5 \end{array} \end{array}$$

$\vec{v}_2 = 2\vec{v}_1 \Longleftarrow \vec{w}_2 = 2\vec{w}_1$ $\quad \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\vec{v}_1, \vec{v}_3$ are ind. $\Longleftarrow$ $\vec{w}_1, \vec{w}_3$ are independent

$\vec{v}_4 = \vec{v}_1 + 2\vec{v}_3 \Longleftarrow \vec{w}_4 = \vec{w}_1 + 2\vec{w}_3$ $\quad \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\vec{v}_5 = \vec{v}_1 - \vec{v}_3 \Longleftarrow \vec{w}_5 = \vec{w}_1 - \vec{w}_3$

compare to previous page

Exercise 3) (This exercise explains why each matrix has only one reduced row echelon form, no matter what sequence of elementary row operations one uses to find it. We didn't have the tools to explain why this fact was true earlier in the chapter) Let $B_{4\times 5}$ be a matrix whose columns satisfy the following dependencies:

$$col_1(B) \neq \underline{\mathbf{0}} \quad \text{(i.e. is independent)}$$
$$col_2(B) = 3\, col_1(B)$$
$$col_3(B) \text{ is independent of column 1}$$
$$col_4(B) \text{ is independent of columns 1,3.}$$
$$col_5(B) = -3\, col_1(B) + 2\, col_3(B) - col_4(B).$$

What is the reduced row echelon form of B?

B 4 rows, 5 columns

$$B = \left[\vec{b}_1\ \vec{b}_2\ \vec{b}_3\ \vec{b}_4\ \vec{b}_5\right] \Longrightarrow rref(B) = \begin{bmatrix} 1 & 3 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$-3\begin{bmatrix}1\\0\\0\\0\end{bmatrix} + 2\begin{bmatrix}0\\1\\0\\0\end{bmatrix} - 1\begin{bmatrix}0\\0\\1\\0\end{bmatrix}$$

What if:
$$\begin{matrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}$$

would violate rref conditions!

depending on 5th column would either violate

* zero rows at bottom

or * pivots move to right as you go down the rows

1.8 Introduction to linear transformations.

F

Definition: A function T which has domain equal to $\mathbb{R}^n$ and whose range lies in $\mathbb{R}^m$ is called a *linear transformation* if it transforms sums to sums, and scalar multiples to scalar multiples. Precisely, $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is *linear* if and only if

$$T(\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}}) = T(\underline{\boldsymbol{u}}) + T(\underline{\boldsymbol{v}}) \qquad \forall\ \underline{\boldsymbol{u}}, \underline{\boldsymbol{v}} \in \mathbb{R}^n$$
$$T(c\,\underline{\boldsymbol{u}}) = c\,T(\underline{\boldsymbol{u}}) \qquad \forall\ c \in \mathbb{R}, \underline{\boldsymbol{u}} \in \mathbb{R}^n\ .$$

Notation In this case we call $\mathbb{R}^m$ the *codomain*. We call $T(\underline{\boldsymbol{u}})$ the *image of* $\underline{\boldsymbol{u}}$. The *range* of T is the collection of all images $T(\underline{\boldsymbol{u}})$, for $\underline{\boldsymbol{u}} \in \mathbb{R}^n$.

Important connection to matrices: Each matrix $A_{m \times n}$ gives rise to a linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$, namely

$$T(\underline{\boldsymbol{x}}) := A\,\underline{\boldsymbol{x}} \qquad \forall\ \underline{\boldsymbol{x}} \in \mathbb{R}^n.$$

e.g.

$$\begin{bmatrix} -1 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \qquad \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

This is because

Theorem 5 (p. 39 Matrix multiplication is linear) If A is an $m \times n$ matrix, $\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}} \in \mathbb{R}^n$, c a scalar, then

a) $A(\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}}) = A\,\underline{\boldsymbol{u}} + A\,\underline{\boldsymbol{v}}$

b) $A(c\,\underline{\boldsymbol{u}}) = c\,A\,\underline{\boldsymbol{u}}$

e.g. n=3 columns

$$A(\vec{u}+\vec{v}) = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{bmatrix} \begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ u_3+v_3 \end{bmatrix} = (u_1+v_1)\vec{a}_1 + (u_2+v_2)\vec{a}_2 + (u_3+v_3)\vec{a}_3$$
$$= u_1\vec{a}_1 + u_2\vec{a}_2 + u_3\vec{a}_3$$
$$+ v_1\vec{a}_1 + v_2\vec{a}_2 + v_3\vec{a}_3$$
$$= A\vec{u} + A\vec{v}$$

Remark: One reason that the word "linear" is appropriate for these sorts of functions is that linear transformations transform lines to lines (or points); and families of parallel lines are transformed into families of parallel lines (or points).

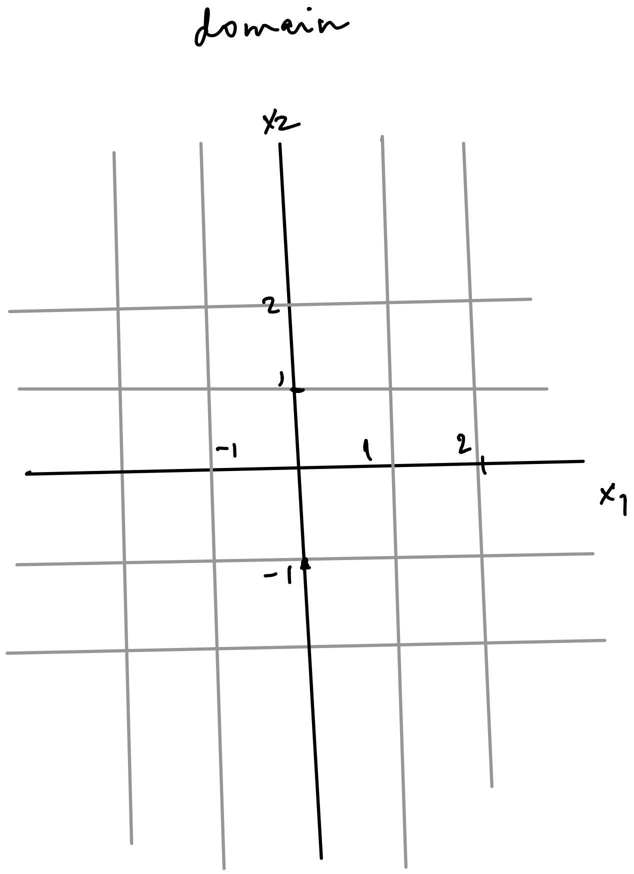
to be continued!

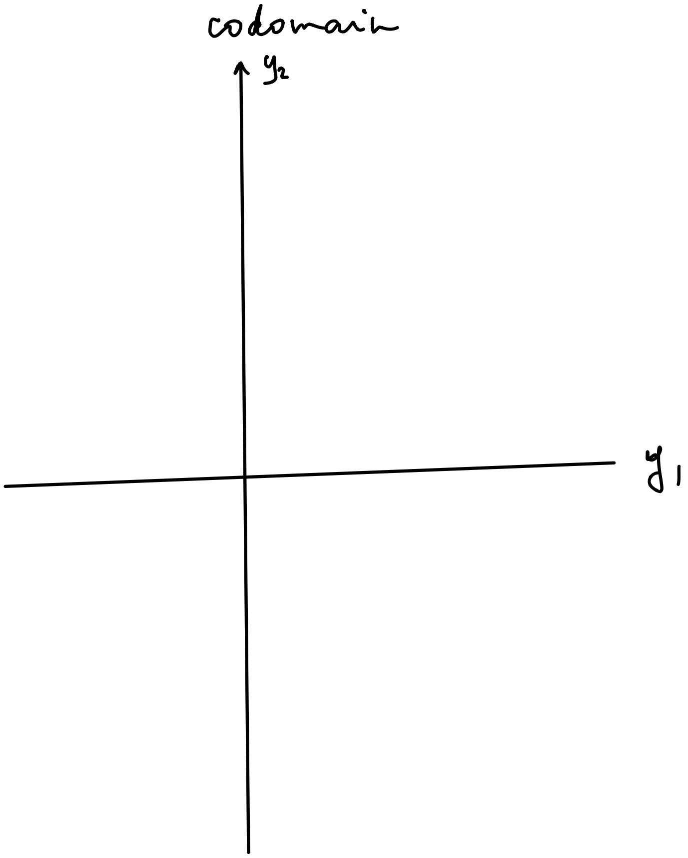
Exercise 4) Let $T:\mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) := \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A\,\underline{\boldsymbol{x}}.$$

4a Find $T(\underline{\boldsymbol{u}})$ for $\underline{\boldsymbol{u}} = \underline{\boldsymbol{e}}_1, \underline{\boldsymbol{e}}_2$. What parts of the matrix do these output vectors correspond to?

4b) How is the domain rectangular grid generated by the standard basis vectors transformed by this function T ?





4c) Consider domain lines with slope 1, i.e. with direction vectors parallel to $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. What are the slopes of the images of these lines ?

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) := \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A\,\underline{\boldsymbol{x}}$$

4d) Sketch the image of the line segment having position vectors $\left\{ t\begin{bmatrix} 1 \\ 1 \end{bmatrix} \ s.t. \ -1 \le t \le 2 \right\}$.