Math 2270-002 Week 2 notes

We will not necessarily finish the material from a given day's notes on that day. We may also add or subtract some material as the week progresses, but these notes represent an in-depth outline of what we plan to cover. These notes cover material in 1.3-1.6. They include material from last weeks notes that we did not get to.

1.3 vector equations

1.4 matrix equations encompass vector equations and linear systems of equations

1.5 structure of solution sets to matrix equations

1.6 some applications

Mon Aug 27

- 1.3 algebra and geometry for vector equations and linear combinations

Announcements:

- returned quizzes are in folders (soltns on CANVAS)
- new notes for week 2
- M, T office hours request here LCB 215 → hour after class (T class LCB 225 not available)

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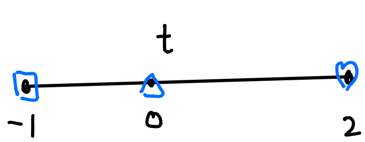
Warm-up Exercise: Sketch the line segment of points, whose position vectors are given by

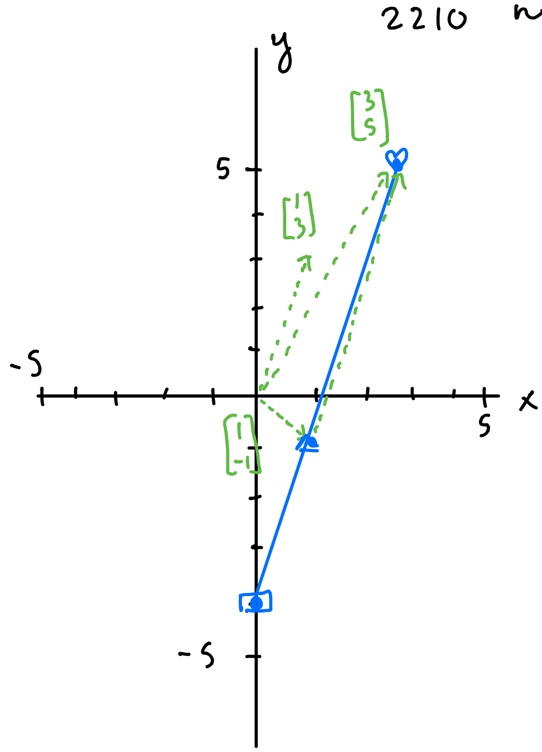
$$\vec{r}(t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} + t\begin{bmatrix} 1 \\ 3 \end{bmatrix} = (1+t)\hat{i} + (-1+3t)\hat{j} = \underbrace{\langle 1+t, -1+3t \rangle}_{\text{2210 notation}}$$

for $-1 \le t \le 2$

or $x = 1+t$, $y = -1+3t$, $-1 \le t \le 2$

t	$\vec{r}(t)$	(x,y)	
0	$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$	$(1,-1)$	
-1	$\begin{bmatrix} 0 \\ -4 \end{bmatrix}$	$(0,-4)$	$\begin{bmatrix} 1 \\ -1 \end{bmatrix} - 1\begin{bmatrix} 1 \\ 3 \end{bmatrix}$
2	$\begin{bmatrix} 3 \\ 5 \end{bmatrix}$	$(3,5)$	$\begin{bmatrix} 1 \\ -1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 3 \end{bmatrix}$





On Friday we defined vectors algebraically, as ordered lists of numbers. And, we defined vector addition and scalar multiplication, which you've worked with in previous courses, although maybe only in $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3$:

$$\underline{\text{Definition:}} \text{ For } \underline{\boldsymbol{u}} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \underline{\boldsymbol{v}} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n; c \in \mathbb{R}, \text{ then } \quad \underline{\boldsymbol{u}} + \underline{\boldsymbol{v}} := \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix}; \ c\,\underline{\boldsymbol{u}} := \begin{bmatrix} cu_1 \\ cu_2 \\ \vdots \\ cu_n \end{bmatrix}.$$

There are a number of straightforward algebra identities for vector addition and scalar multiplication. These all reduce to real number axioms when one looks at the individual entries of the vectors on each side of the identities:

Let $\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}}, \underline{\boldsymbol{w}} \in \mathbb{R}^n, \quad c, d \in \mathbb{R}$. Then

(i) $\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}} = \underline{\boldsymbol{v}} + \underline{\boldsymbol{u}}$

e.g. $\text{entry}_i(\vec{u}+\vec{v}) = u_i + v_i$
$\text{entry}_i(\vec{v}+\vec{u}) = v_i + u_i$ $\Big\}$ equal because real # addition is commutative

(ii) $(\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}}) + \underline{\boldsymbol{w}} = \underline{\boldsymbol{u}} + (\underline{\boldsymbol{v}} + \underline{\boldsymbol{w}})$

$\text{entry}_i((\vec{u}+\vec{v})+\vec{w}) = (u_i+v_i)+w_i$
$\text{entry}_i(\vec{u}+(\vec{v}+\vec{w})) = u_i + (v_i+w_i)$ $\Big\}$ equal because real # addition is associative

(iii) $\underline{\boldsymbol{u}} + \underline{\boldsymbol{0}} = \underline{\boldsymbol{0}} + \underline{\boldsymbol{u}} = \underline{\boldsymbol{u}}$ ($\underline{\boldsymbol{0}}$ is defined to be the vector for which each entry is the number 0.)

(iv) $\underline{\boldsymbol{u}} + (-\underline{\boldsymbol{u}}) = (-\underline{\boldsymbol{u}}) + \underline{\boldsymbol{u}} = \underline{\boldsymbol{0}}$ ($-\underline{\boldsymbol{u}}$ is defined to be $-1 \cdot \underline{\boldsymbol{u}}$, i.e. the vector for which each entry is the opposite of the corresponding entry in $\underline{\boldsymbol{u}}$.)

(v) $c(\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}}) = c\,\underline{\boldsymbol{u}} + c\,\underline{\boldsymbol{v}}$

(vi) $(c+d)\underline{\boldsymbol{u}} = c\,\underline{\boldsymbol{u}} + d\,\underline{\boldsymbol{u}}$

(vii) $c(d\,\underline{\boldsymbol{u}}) = (c\,d)\,\underline{\boldsymbol{u}}$

(viii) $1\,\underline{\boldsymbol{u}} = \underline{\boldsymbol{u}}$.

Geometric interpretation of vectors as displacements

The space $\mathbb{R}^n$ may be thought of in two equivalent ways. In both cases, $\mathbb{R}^n$ consists of all possible $n-$ *tuples* of numbers:

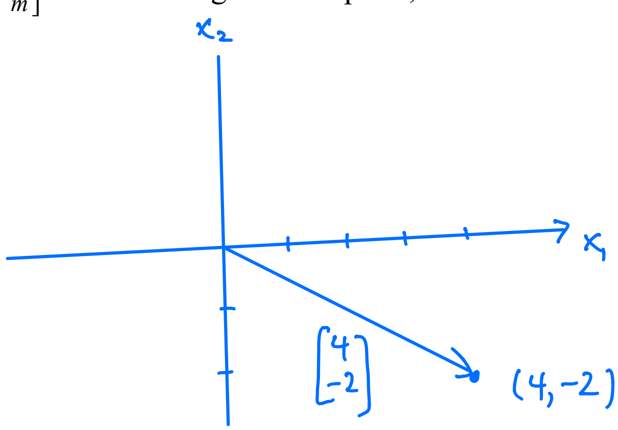
(i) We can think of those $n-$ *tuples* as representing points, as we're used to doing for $n = 1, 2, 3$. In this case we can write

$$\mathbb{R}^n = \{(x_1, x_2, ..., x_n), s.t.\ x_1, x_2, ..., x_n \in \mathbb{R}\}.$$

(ii) We can think of those $n-$ *tuples* as representing vectors that we can add and scalar multiply. In this case we can write

$$\mathbb{R}^n = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ : \\ x_n \end{bmatrix}, s.t.\ x_1, x_2, ..., x_n \in \mathbb{R} \right\}.$$

Since algebraic vectors (as above) can be used to measure geometric displacement, one can identify the two models of $\mathbb{R}^n$ as sets by identifying each point $(x_1, x_2, ...x_n)$ in the first model with the displacement vector $\underline{\mathbf{x}} = [x_1, x_2, ...x_m]^T$ from the origin to that point, in the second model, i.e. the "position vector" of the point.



Exercise 1 : Finish this exercise from Friday ...

Exercise 3) Let $\underline{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\underline{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.
(Friday)

3a) Plot the points $(1, -1)$ and $(1, 3)$, which have position vectors $\underline{u}$, $\underline{v}$. Draw these position vectors as arrows beginning at the origin and ending at the corresponding points.

3b) Compute $\underline{u} + \underline{v}$ and then plot the point for which this is the position vector. Note that the algebraic operation of vector addition corresponds to the geometric process of composing horizontal and vertical displacements.

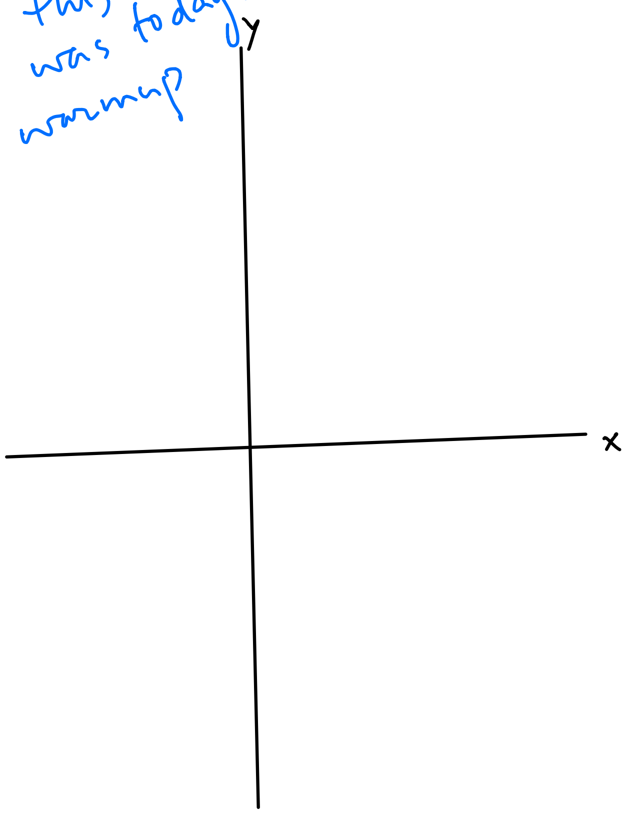
$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

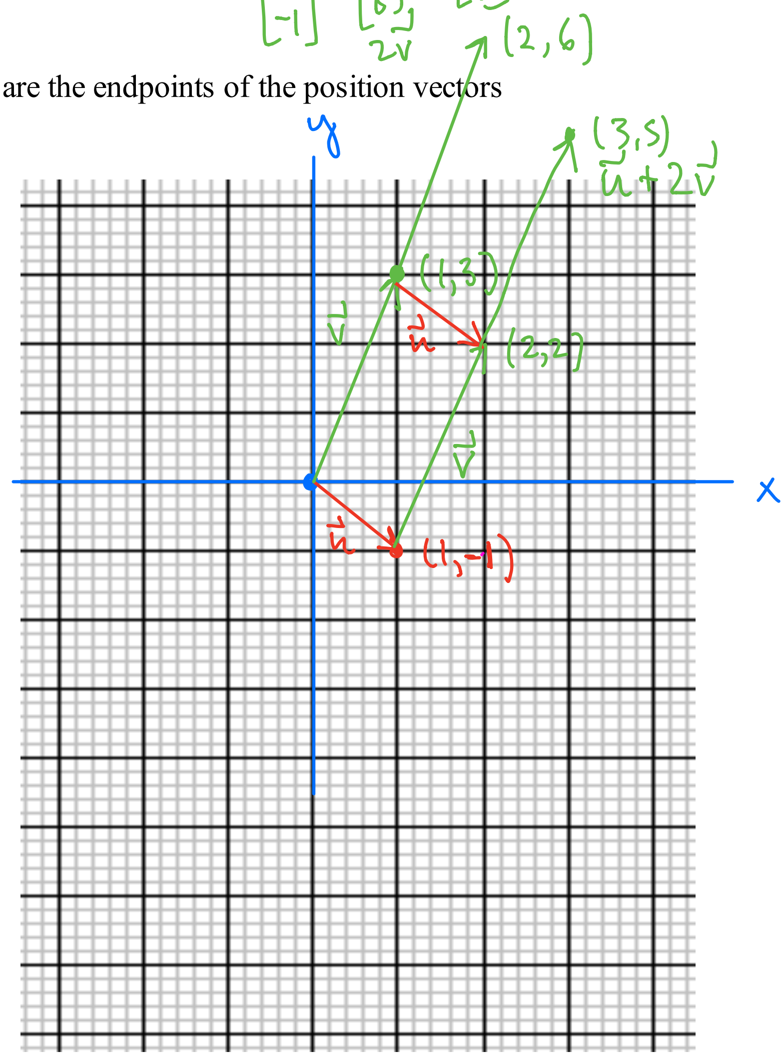
3c) Compute $\underline{u}$ and $2\,\underline{v}$ and $\underline{u} + 2\,\underline{v}$ plot the corresponding points for which these are the position vectors.

$$2\vec{v} = 2\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix} \qquad \vec{u} + 2\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \underbrace{\begin{bmatrix} 2 \\ 6 \end{bmatrix}}_{2\vec{v}} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

3d) Plot the parametric line segment whose points are the endpoints of the position vectors $\{\underline{u} + t\,\underline{v},\ -1 \le t \le 2\}$.

this was today's warmup





One of the key themes of this course is the idea of "linear combinations". These have an algebraic definition, as well as a geometric interpretation as combinations of displacements.

Definition: If we have a collection of n vectors $\{\underline{v}_1, \underline{v}_2, \ldots \underline{v}_p\}$ in $\mathbb{R}^n$, then any vector $\underline{v} \in \mathbb{R}^n$ that can be expressed as a sum of scalar multiples of these vectors is called a *linear combination* of them. In other words, if we can write

$$\underline{v} = c_1\underline{v}_1 + c_2\underline{v}_2 + \ldots + c_p\underline{v}_p ,$$

then $\underline{v}$ is a *linear combination* of $\underline{v}_1, \underline{v}_2, \ldots \underline{v}_p$. The scalars $c_1, c_2, \ldots, c_p$ are called the *linear combination coefficients* or *weights*.

Example You've probably seen linear combinations in previous math/physics classes, even if you didn't realize it. For example you might have expressed the position vector $\underline{\mathbf{r}}$ as a linear combination

$$\underline{\mathbf{r}} = x\,\underline{\mathbf{i}} + y\,\underline{\mathbf{j}} + z\,\underline{\boldsymbol{k}} \quad = \langle x, y, z \rangle$$

where $\underline{\mathbf{i}}, \underline{\mathbf{j}}, \underline{\mathbf{k}}$ represent the unit displacements in the x, y, z directions. Since we can express these displacements using Math 2270 notation as

$$\underline{\mathbf{i}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ \underline{\mathbf{j}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \ \underline{\mathbf{k}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

we have

$$x\,\underline{\mathbf{i}} + y\,\underline{\mathbf{j}} + z\,\underline{\boldsymbol{k}} = x\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Exercise 2) Can you get to the point $(-2, 8) \in \mathbb{R}^2$, from the origin $(0, 0)$, by moving only in the $(\pm)$ directions of $\underline{\boldsymbol{u}} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\underline{\boldsymbol{v}} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$? Algebraically, this means we want to solve the linear combination problem

$$x_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 8 \end{bmatrix}.$$

2a) Superimpose a grid related to the displacement vectors $\underline{\boldsymbol{u}}$, $\underline{\boldsymbol{v}}$ onto the graph paper below, and, recalling that vector addition yields net displacement, and scalar multiplication yields scaled displacement, try to approximately solve the linear combination problem above, geometrically.

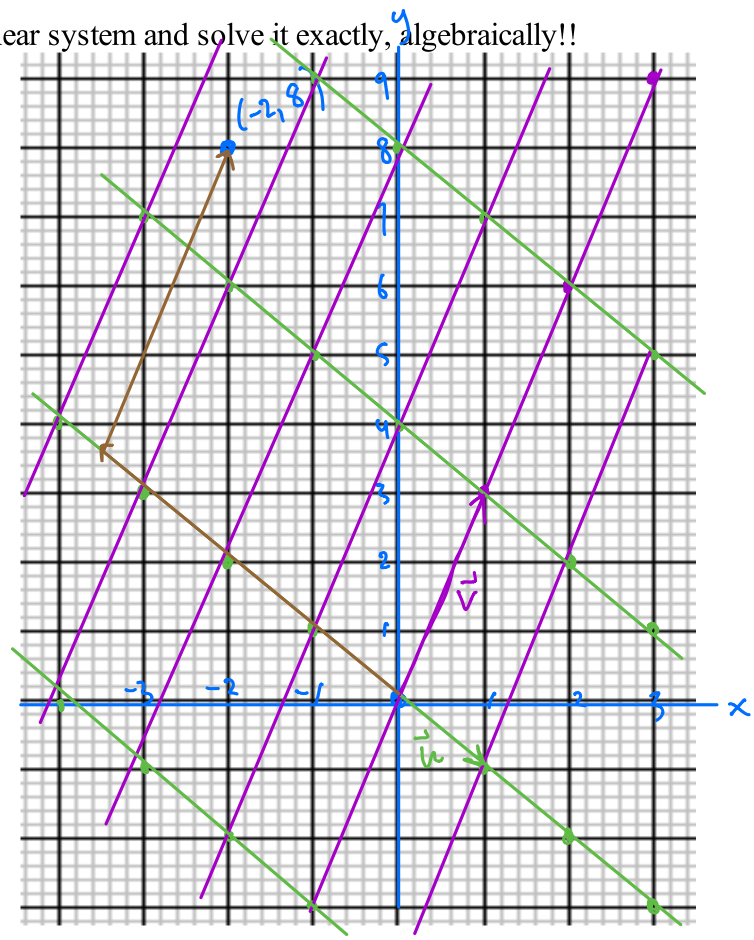
2b) Rewrite the linear combination problem as a linear system and solve it exactly, algebraically!!

geometric guess

$$\begin{bmatrix} -2 \\ 8 \end{bmatrix} \approx -3.5\,\vec{u} + 1.5\,\vec{v}$$

$$\stackrel{?}{=} -3.5 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + 1.5 \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} -2 \\ 8 \end{bmatrix} \; !$$



$$x_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ -x_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ 3x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 8 \end{bmatrix}$$

$$\begin{aligned} x_1 + x_2 &= -2 \\ -x_1 + 3x_2 &= 8 \end{aligned}$$

Lin Sys!

$$\left[\begin{array}{cc|c} 1 & 1 & -2 \\ -1 & 3 & 8 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 1 & -2 \\ 0 & 4 & 6 \end{array}\right] \quad R_1 + R_2 \rightarrow R_2$$

$$\left[\begin{array}{cc|c} 1 & 1 & -2 \\ 0 & 1 & 1.5 \end{array}\right] \quad R_2/4 \rightarrow R_2$$

$$\left[\begin{array}{cc|c} 1 & 0 & -3.5 \\ 0 & 1 & 1.5 \end{array}\right] \quad -R_2 + R_1 \rightarrow R_1$$

algebraically, $x_1 = -3.5$
$x_2 = 1.5$

agrees.

2c) Can you get to any point (x, y) in $\mathbb{R}^2$, starting at $(0, 0)$ and moving only in directions parallel to $\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}}$?

Argue geometrically and algebraically. How many ways are there to express $\begin{bmatrix} x \\ y \end{bmatrix}$ as a linear combination of $\underline{\boldsymbol{u}}$ and $\underline{\boldsymbol{v}}$?

geometric reason: $\vec{u}, \vec{v}$ are not parallel, so the "$\vec{u}, \vec{v}$" grid covers all of $\mathbb{R}^2$.

algebraic reason:

instead of $(-2, 8)$

vector eqtn is $x_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$

augmented matrix $\left[\begin{array}{cc|c} 1 & 1 & x \\ -1 & 3 & y \end{array}\right]$ reduces to $\left[\begin{array}{cc|c} 1 & 0 & c_1 \\ 0 & 1 & c_2 \end{array}\right]$

so $x_1 = c_1$
$x_2 = c_2$

made out of x's & y's.

Definition The *span* of a collection of vectors, written as $span\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n\}$, is the collection of all linear combinations of those vectors.

Examples using this language:

$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

- We showed in 2c that $span\{\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}}\} = \mathbb{R}^2$

such that

$span\{\vec{u}, \vec{v}\} := \{\vec{w} = c_1\vec{v}_1 + c_2\vec{v}_2 \text{ s.t. } c_1, c_2 \in \mathbb{R}\}$

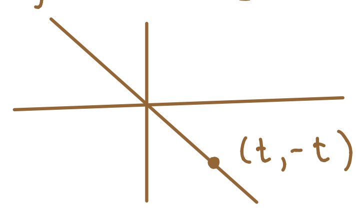
"is defined to be"

- The other hand, $span\{\underline{\boldsymbol{u}}\}$ is the line with implicit equation $y = -x$.

$span\{\vec{u}\} := \{c_1\vec{u} \text{ s.t. } c_1 \in \mathbb{R}\} = \{t \begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ s.t. } t \in \mathbb{R}\}$

unlike "wingspan", vector "span" continues indefinitely

- in $\mathbb{R}^3$, $span\{\underline{\boldsymbol{i}}, \underline{\boldsymbol{j}}, \underline{\boldsymbol{k}}\} = \mathbb{R}^3$.



pt. with pos. vector $\begin{bmatrix} t \\ -t \end{bmatrix}$

Remark: The mathematical meaning of the word *span* is related to the English meaning - as in "wing span" or "span of a bridge", but it's also different. The span of a collection of vectors goes on and on and does not "stop" at the vector or associated endpoint.

What we may have realized in the previous exercise is the very important:

Fundamental Fact A *vector equation* (linear combination problem)

$$x_1\,\underline{\boldsymbol{a}}_1 + x_2\,\underline{\boldsymbol{a}}_2 + \ldots + x_n\,\underline{\boldsymbol{a}}_n = \underline{\boldsymbol{b}} \qquad \text{in } \mathbb{R}^m$$

is equivalent to a system of linear equations for the unknown weights $x_1, x_2, \ldots\ x_n$; in fact the system of linear equations has augmented matrix given by

$$\left[\underline{\boldsymbol{a}}_1 \ \ \underline{\boldsymbol{a}}_2 \ \cdots \ \underline{\boldsymbol{a}}_n \ \ \underline{\boldsymbol{b}}\ \right]$$

(where we have expressed the augmented matrix in terms of its columns). In particular, $\underline{\boldsymbol{b}}$ can be generated by a linear combination of $\underline{\boldsymbol{a}}_1 \ \ \underline{\boldsymbol{a}}_2 \ \ldots \ \underline{\boldsymbol{a}}_n$ if and only if there exists a solution to the linear system corresponding to the augmented matrix above. Once we recognize the equivalence we can answer any question about a linear combination vector equation using Gaussian elimination and reduced row echelon form computations and concepts.

This fundamental fact is so important to the course, that we should check it in general at some point.

$$= x_1\begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \vdots \\ a_{m1}\end{bmatrix} + x_2\begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2}\end{bmatrix} + \ldots + x_n\begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn}\end{bmatrix} = \begin{bmatrix} x_1 a_{11} \\ x_1 a_{21} \\ \vdots \\ x_1 a_{m1}\end{bmatrix} + \begin{bmatrix} x_2 a_{12} \\ x_2 a_{22} \\ \vdots \\ x_2 a_{m2}\end{bmatrix} + \ldots + \begin{bmatrix} x_n a_{1n} \\ x_n a_{2n} \\ \vdots \\ x_n a_{mn}\end{bmatrix}$$

§1.1-1.2 linear system:

$$= \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix} \overset{\text{want}}{=} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Exercise 3a) Does the vector equation

$$x_1\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + x_2\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$$

have any solutions?

we did this in warmup $x_1 = 1/2$, $x_2 = -3/2$

3b) What geometric question is this related to? What geometric object is $span\left\{\underbrace{\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}}_{\text{expect plane}}\right\}$?

3c) Use an augmented matrix calculation to find what condition needs to hold on vectors $\underline{\boldsymbol{b}}$ so that $\underline{\boldsymbol{b}} \in span\left\{\begin{bmatrix}1\\0\\2\end{bmatrix}, \begin{bmatrix}-1\\2\\0\end{bmatrix}\right\}$. (!!) How does this computation relate to the (implicit) way we've been expressing planes in $\mathbb{R}^3$?

for which $\vec{b}$??

to see when we can solve

$$x_1\begin{bmatrix}1\\0\\2\end{bmatrix} + x_2\begin{bmatrix}-1\\2\\0\end{bmatrix} = \begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & -1 & b_1 \\ 0 & 2 & b_2 \\ 2 & 0 & b_3 \end{array}\right]$$

$$\begin{array}{l} R_3/2 \to R_1 \\ R_2/2 \to R_2 \\ R_1 \end{array} \left[\begin{array}{cc|c} 1 & 0 & b_3/2 \\ 0 & 1 & b_2/2 \\ 1 & -1 & b_1 \end{array}\right]$$

$$\begin{array}{l} \\ \\ -R_1+R_3 \to R_3 \end{array} \left[\begin{array}{cc|c} 1 & 0 & b_3/2 \\ 0 & 1 & b_2/2 \\ 0 & -1 & b_1 - b_3/2 \end{array}\right]$$

$$\begin{array}{l} \\ \\ R_2+R_3 \to R_3 \end{array} \left[\begin{array}{cc|c} 1 & 0 & b_3/2 \\ 0 & 1 & b_2/2 \\ 0 & 0 & b_1 - b_3/2 + b_2/2 \end{array}\right]$$

$$\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix} = \begin{bmatrix}x\\y\\z\end{bmatrix}$$

consistent

$$b_1 - \frac{b_3}{2} + \frac{b_2}{2} = 0$$

$$x - \frac{z}{2} + \frac{y}{2} = 0$$

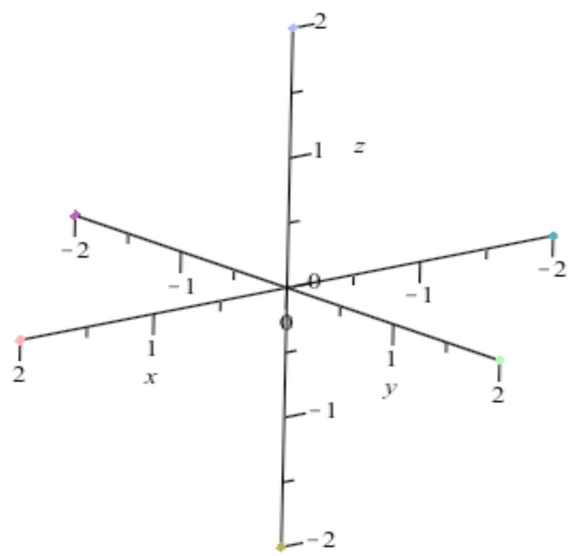
$$\boxed{2x + y - z = 0}$$

also $x_1 = b_3/2$, $x_2 = b_2/2$

$$\begin{bmatrix}2\\-3\\1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\0\\2\end{bmatrix} - \frac{3}{2}\begin{bmatrix}-1\\2\\0\end{bmatrix}$$

inconsistent if $2x + y - z \neq 0$

In case we want to sketch anything related to Exercise 3:



Tues Aug 28

- 1.4 the matrix equation $A\,\underline{\mathbf{x}} = \underline{\boldsymbol{b}}$. How the reduced row echelon form of (just) A relates to solvability questions (leads into section 1.5).

<u>Announcements:</u>

- MTW 2-2:50 LCB 225 office hours (I need to change on syllabus)
- HW due tomorrow
- Quiz as well 1.1-1.3

'til 12:57

<u>Warm-up Exercise</u>: Is $\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$ in $\text{span}\left\{ \overset{\vec{u}}{\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}}, \overset{\vec{v}}{\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}} \right\}$? (Exercise 3a yesterday's notes)

Hint: the vector equation you're trying to solve is

$$* \qquad x_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$$

yes: $\frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - \frac{3}{2}\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$ ✓

How? $*$ vector eqtn is equiv. to linear sys. with augmented matrix

$$\left[\begin{array}{cc|c} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 2 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 1/2 \\ 0 & 1 & -3/2 \\ 0 & 0 & 0 \end{array}\right]$$

terminology:

<u>span</u> $\{\vec{u}, \vec{v}\}$:= collection of all <u>linear combinations of $\vec{u}, \vec{v}$</u>

any $\vec{w} = c_1\vec{u} + c_2\vec{v} \quad c_1, c_2 \in \mathbb{R}$

(or $x_1\vec{u} + x_2\vec{v}$)

Recall ✓

Fundamental Fact A *vector equation* (linear combination problem)

$$x_1 \underline{\boldsymbol{a}}_1 + x_2 \underline{\boldsymbol{a}}_2 + \ \dots \ + x_n \underline{\boldsymbol{a}}_n = \underline{\boldsymbol{b}}$$

is actually a system of linear equations for the unknown weights $x_1, x_2, \dots x_n$; in fact the system of linear equations has augmented matrix given by

$$\left[\underline{\boldsymbol{a}}_1 \ \ \underline{\boldsymbol{a}}_2 \ \cdots \ \underline{\boldsymbol{a}}_n \ \underline{\boldsymbol{b}} \ \right]$$

(where we have expressed the augmented matrix in terms of its columns). In particular, $\underline{\boldsymbol{b}}$ can be generated by a linear combination of $\underline{\boldsymbol{a}}_1 \ \ \underline{\boldsymbol{a}}_2 \ \dots \ \underline{\boldsymbol{a}}_n$ if and only if there exists a solution to the linear system corresponding to the augmented matrix above.

We should check this carefully today, assuming we didn't do so on Monday:

Definition (from 1.4) If A is an $m \times n$ matrix, with columns $\underline{\boldsymbol{a}}_1, \underline{\boldsymbol{a}}_2, \dots \underline{\boldsymbol{a}}_n$ (in $\mathbb{R}^m$) and if $\underline{\boldsymbol{x}} \in \mathbb{R}^n$, then $A\,\underline{\boldsymbol{x}}$ is defined to be the linear combination of the columns, with weights given by the corresponding entries of $\underline{\boldsymbol{x}}$. In other words,

$$A\,\underline{\boldsymbol{x}} := x_1 \underline{\boldsymbol{a}}_1 + x_2 \underline{\boldsymbol{a}}_2 + \ \dots x_n \underline{\boldsymbol{a}}_n \ .$$

(This will give us a way to abbreviate vector equations, for example.)

$$\begin{bmatrix} 1 & 3 & -6 \\ 2 & 4 & 17 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 4 \end{bmatrix} + 5\begin{bmatrix} -6 \\ 17 \end{bmatrix}$$

Definition. Let $\underline{u}, \underline{v}$ be vectors in $\mathbb{R}^n$. Then the *dot product* $\underline{u} \cdot \underline{v}$ is defined by

$$\underline{u} \cdot \underline{v} = \sum_{j=1}^{n} u_j\, v_j = u_1\, v_1 + u_2\, v_2 + \dots u_n\, v_n\,.$$

$$\begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -1 \\ 6 \end{bmatrix} = 1\cdot 3 + 4\cdot(-1) + (-2)\cdot 6$$
$$= 3 - 4 - 12 = -13$$

Computational Theorem: (This is usually a quicker way to compute $A\,\underline{x}$. Let If A be an $m \times n$ matrix, with rows $R_1, R_2, \dots R_m$. Then $A\,\underline{x}$ may also be computed using the rows of A and the dot product:

$$x_1\, \underline{a}_1 + x_2\, \underline{a}_2 + \dots x_n\, \underline{a}_n \overset{\text{def}}{=} A\,\underline{x} \overset{\text{OR}}{=} \begin{bmatrix} R_1 \cdot \underline{x} \\ R_2 \cdot \underline{x} \\ \vdots \\ R_m \cdot \underline{x} \end{bmatrix}$$

go look at the fact: vector eqtns = linear systems

Exercise 1a) Compute both ways:

$$\begin{bmatrix} 1 & -2 & 3 \\ -2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = 2\begin{bmatrix} 1 \\ -2 \end{bmatrix} - 2\begin{bmatrix} -2 \\ 3 \end{bmatrix} + 1\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \end{bmatrix} + \begin{bmatrix} 4 \\ -6 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 9 \\ -6 \end{bmatrix}$$

OR

$$= \begin{bmatrix} 1\cdot 2 + (-2)(-2) + 3\cdot 1 \\ (-2)(2) + 3(-2) + 4\cdot 1 \end{bmatrix} = \begin{bmatrix} 9 \\ -6 \end{bmatrix}$$

Exercise 1b) Write as a matrix times a vector:

$$3\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} + 2\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 & 2 & -1 \\ 1 & 3 & 2 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}$$

Exercise 2) Rewrite the following vector equations from yesterday and last week as matrix equations. Also write down the augmented matrix for these systems.

2a)

$$x_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 8 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 1 & -2 \\ -1 & 3 & 8 \end{array}\right]$$

2b)

$$x_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + x_1 \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$$

What the solvability and number of solutions to a matrix equation $A\,\underline{\mathbf{x}} = \underline{\boldsymbol{b}}$ has to do with the reduced row echelon form of A (i.e. of the unaugmented matrix). Let's explore.

Exercise 3 Find all solutions to the system of 3 linear equations in 5 unknowns

(skip in class... this is review of algorithm for rref.)

$$\begin{aligned} x_1 - 2x_2 + 3x_3 + 2x_4 + x_5 &= 10 \\ 2x_1 - 4x_2 + 8x_3 + 3x_4 + 10x_5 &= 7 \\ 3x_1 - 6x_2 + 10x_3 + 6x_4 + 5x_5 &= 27\,. \end{aligned}$$

Here's the augmented matrix:

$$\left[\begin{array}{rrrrr|r} 1 & -2 & 3 & 2 & 1 & 10 \\ 2 & -4 & 8 & 3 & 10 & 7 \\ 3 & -6 & 10 & 6 & 5 & 27 \end{array}\right]$$

Find the reduced row echelon form of this augmented matrix and then backsolve to explicitly parameterize the solution set. (Hint: it's a two-dimensional plane in $\mathbb{R}^5$, if that helps. :-))

Maple says:

```
> with(LinearAlgebra) :   # matrix and linear algebra library
> A := Matrix(3, 5, [1, -2, 3, 2, 1,
                     2, -4, 8, 3, 10,
                     3, -6, 10, 6, 5]) :
  b := Vector([10, 7, 27]) :
  <A|b>;     # the mathematical augmented matrix doesn't actually have
             # a vertical line between the end of A and the start of b
  ReducedRowEchelonForm(<A|b>);
```

$$\begin{bmatrix} 1 & -2 & 3 & 2 & 1 & 10 \\ 2 & -4 & 8 & 3 & 10 & 7 \\ 3 & -6 & 10 & 6 & 5 & 27 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 0 & 0 & 3 & 5 \\ 0 & 0 & 1 & 0 & 2 & -3 \\ 0 & 0 & 0 & 1 & -4 & 7 \end{bmatrix} \quad \textbf{(1)}$$

```
> LinearSolve(A, b);
      #  this command will actually write down the general solution, using
      # Maple's way of writing free parameters, which actually makes
      # some sense.  Generally when there  are free parameters involved,
      #  there will be equivalent ways to express the solution that may
      # look different.  But usually Maple's version will look like yours,
      # because it's using the same algorithm and choosing the free parameters
      # the same way too.
```

$$\begin{bmatrix} 5 + 2\_t_2 - 3\_t_5 \\ \_t_2 \\ -3 - 2\_t_5 \\ 7 + 4\_t_5 \\ \_t_5 \end{bmatrix} \quad \textbf{(2)}$$

```
>
```

In HW, you're thinking abt: ① what makes linear sys consistent / inconsistent
② when do you get 1 or ∞'ly sol's for consistent systems

Exercise 4 We are interested in the matrix equation $A\,\underline{\mathbf{x}} = \underline{\boldsymbol{b}}$ for the matrix A below, and three different right hand sides at once.

$$
\begin{array}{cccccc}
\text{vars} & x_1 & x_2 & x_3 & x_4 & x_5
\end{array}
$$

$$
A := \begin{bmatrix} 2 & 7 & -10 & -19 & 13 \\ 1 & 3 & -4 & -8 & 6 \\ 1 & 0 & 2 & 1 & 3 \end{bmatrix} \qquad rref(A) = \begin{bmatrix} 1 & 0 & 2 & 1 & 3 \\ 0 & 1 & -2 & -3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.
$$

Let's consider three different linear systems for which A is the coefficient matrix. In the first one, the right hand sides are all zero (what we call the "homogeneous" problem), and I have carefully picked the other two right hand sides. The three right hand sides are separated by the dividing line below:

$$
C := \left[\begin{array}{ccccc|ccc} 2 & 7 & -10 & -19 & 13 & 0 & 7 & 7 \\ 1 & 3 & -4 & -8 & 6 & 0 & 0 & 3 \\ 1 & 0 & 2 & 1 & 3 & 0 & 0 & 0 \end{array}\right] \qquad rref(C) = \left[\begin{array}{ccccc|c|c|c} 1 & 0 & 2 & 1 & 3 & 0 & 0 & 0 \\ 0 & 1 & -2 & -3 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]
$$

4a) Find the solution sets for each of the three systems, using the reduced row echelon form of C.

sys. 1 $\vec{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\qquad \vec{b} = \begin{bmatrix} 7 \\ 0 \\ 0 \end{bmatrix}$ $\qquad \vec{b} = \begin{bmatrix} 7 \\ 3 \\ 0 \end{bmatrix}$

sys 1

$x_1 + 2x_3 + x_4 + 3x_5 = 0$

$x_2 - 2x_3 - 3x_4 + x_5 = 0$

syst. 1

$x_1 = -2t_3 - t_4 - 3t_5$

$x_2 = 2t_3 + 3t_4 - t_5$

$x_3 = t_3 \in \mathbb{R}$

$x_4 = t_4 \in \mathbb{R}$

$x_5 = t_5 \in \mathbb{R}$ (free)

syst. 2

inconsistent (no soltns)

3rd eqtn requires $0x_1 + 0x_2 + \dots + 0x_5 = 1$

$x_1 = -2t_3 - t_4 - 3t_5$

$x_2 = 1 + 2t_3 + 3t_4 - t_5$

$x_3 \in t_3 \in \mathbb{R}$

$x_4 = t_4 \in \mathbb{R}$

$x_5 = t_5 \in \mathbb{R}$

$$
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = t_3 \begin{bmatrix} -2 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t_4 \begin{bmatrix} -1 \\ 3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t_5 \begin{bmatrix} -3 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}
$$

same

these solutions are almost the same.

$$
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} +
$$

$$C := \left[\begin{array}{ccccc|c|cc} 2 & 7 & -10 & -19 & 13 & 0 & 7 & 7 \\ 1 & 3 & -4 & -8 & 6 & 0 & 0 & 3 \\ 1 & 0 & 2 & 1 & 3 & 0 & 0 & 0 \end{array}\right] \qquad rref(C) = \left[\begin{array}{ccccc|c|cc} 1 & 0 & 2 & 1 & 3 & 0 & 0 & 0 \\ 0 & 1 & -2 & -3 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Important conceptual questions:

4b) Which of these three solutions on the previous page could you have written down just from the reduced row echelon form of A, i.e. without using the augmented matrix and the reduced row echelon form of the augmented matrix? Why?

1st one, i.e. the homogeneous problem.
because if $\vec{b} = \vec{0}$ then that (augmented) column stays $\vec{0}$ when we do el. row ops

$A\vec{x} = \vec{0}$

4c) Linear systems in which right hand side vectors equal zero are called homogeneous linear systems. Otherwise they are called inhomogeneous or nonhomogeneous. Notice that the general solution to the consistent inhomogeneous system is the sum of a particular solution to it, together with the general solution to the homogeneous system!!! This is a theorem. Can you see why it's true?

later

4d) In general, can you tell how many free parameters the solutions to a matrix system $A\,\underline{\mathbf{x}} = \underline{\mathbf{b}}$ will have, based on the reduced row echelon form of A alone (assuming the system is consistent, i.e. has at least one solution)? State what's true and explain why!

$\underbrace{\text{\# free params}}_{\text{i.e free variables}}$ = # of non-pivot col's in rref (A)

Wed Aug 29

- 1.5 solution sets to matrix equations; homogeneous and nonhomogeneous systems of equations, continued.

Announcements:
- part of next week's hw:
  - 1.4 5, 7, 11, 13, 17, 19, 21, 22, 23, 24, 25, 26, 31
  - 1.5 5, 11, 13, 17, 21, 23
  - more to come ...
- quiz day :)

'til 12:58

Warm-up Exercise:

a) Do the vectors in the set $\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\-1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}\right\}$ span $\mathbb{R}^3$?

i.e. is $\text{span}\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\-1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}\right\} = \mathbb{R}^3$?

you may use the fact that

$$\begin{bmatrix}1 & 4 & 1\\2 & -1 & 0\\3 & 0 & 0\end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}.$$

i.e. can I solve

$$x_1\begin{bmatrix}1\\2\\3\end{bmatrix} + x_2\begin{bmatrix}4\\-1\\0\end{bmatrix} + x_3\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$$

for each $\vec{b} \in \mathbb{R}^3$?

YES

$\xrightarrow{\text{synthetic}}$

$$\left[\begin{array}{ccc|c}1 & 4 & 1 & b_1\\2 & -1 & 0 & b_2\\3 & 0 & 0 & b_3\end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{ccc|c}1 & 0 & 0 & c_1\\0 & 1 & 0 & c_2\\0 & 0 & 1 & c_3\end{array}\right]$$

same el. row ops

c's made out of b_1, b_2, b_3

take $x_1 = c_1$, $x_2 = c_2$, $x_3 = c_3$

b) Do the vectors in the set $\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\-1\\0\end{bmatrix}, \begin{bmatrix}2\\-5\\-6\end{bmatrix}\right\}$ span $\mathbb{R}^3$?

NO

you may use the fact that

$$\begin{bmatrix}1 & 4 & 2\\2 & -1 & -5\\3 & 0 & -6\end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix}1 & 0 & -2\\0 & 1 & 1\\0 & 0 & 0\end{bmatrix}$$

$$x_1\begin{bmatrix}1\\2\\3\end{bmatrix} + x_2\begin{bmatrix}4\\-1\\0\end{bmatrix} + x_3\begin{bmatrix}2\\-5\\-6\end{bmatrix} = \begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$$

$$\left[\begin{array}{ccc|c}1 & 4 & 2 & b_1\\2 & -1 & -5 & b_2\\3 & 0 & -6 & b_3\end{array}\right] \longrightarrow \left[\begin{array}{ccc|c}1 & 0 & -2 & c_1\\0 & 1 & 1 & c_2\\0 & 0 & 0 & c_3\end{array}\right]$$

same el row ops

In fact, the three vectors only spanned a plane in $\mathbb{R}^3$

unless $c_3 = 0$ this system is inconsistent.
(so $\vec{b}$ is not in the span of the set)

so, pick any $\vec{c}$ with $c_3 \neq 0$.
Do reverse ("inverse") elem row ops

$$\left[\begin{array}{ccc|c}1 & 4 & 2 & B_1\\2 & -1 & -5 & B_2\\3 & 0 & -6 & B_3\end{array}\right] \leftarrow \leftarrow \leftarrow \left[\begin{array}{ccc|c}1 & 0 & -2 & c_1\\0 & 1 & 1 & c_2\\0 & 0 & 0 & c_3\end{array}\right]$$

these $\vec{B}$'s are not in the span.

Definition: A system of linear equations is *homogeneous* if it can be written in the form

$$A\,\underline{\mathbf{x}} = \underline{\mathbf{0}}$$

where A is an $m \times n$ matrix, and $\underline{\mathbf{0}}$ is the zero vector in $\mathbb{R}^m$.

Definition: A system of linear equations is *nonhomogeneous* (or inhomogeneous) if it can be written in the form

$$A\,\underline{\mathbf{x}} = \underline{\mathbf{b}}$$

where A is an $m \times n$ matrix, and $\underline{\mathbf{b}}$ is non-zero, i.e. not the zero vector in $\mathbb{R}^m$.

Our goal in section 1.5 is to understand the relationship between the solution sets of homogeneous and nonhomogeneous systems, when the matrix A is the same. And more generally, how the reduced row echeleon form of A is related to the various possibilities.

We've been thinking about the structure of solution sets to systems of linear eqtns...

Friday warm-up exercise, in Wednesday notes

Exercise 1 Consider the matrix A below, and answer all questions (including explanations).

$$A := \left[\begin{array}{ccccc|c} 2 & 7 & -10 & -19 & 13 & 0 \; b_1 \\ 1 & 3 & -4 & -8 & 6 & 0 \; b_2 \end{array}\right] \qquad rref(A) = \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 0 & 2 & 1 & 3 & 0 \; c_1 \\ 0 & 1 & -2 & -3 & 1 & 0 \; c_2 \end{array}\right] \end{array}$$

1a) Is the homogeneous problem $A\mathbf{x}=\mathbf{0}$ always solvable? Can you write down the solutions?

1b) Is the (non) inhomogeneous problem $A\mathbf{x}=\mathbf{b}$ solvable no matter the choice of $\mathbf{b}$? How are the solutions to the nonhomogeneous problems related to those of the homogeneous one?

Yes. Each row of the (reduced) coeff matrix has a pivot. This allows us to backsolve & find solutions.

1c) How many solutions are there? How many free parameters are there in the solution? How does this number relate to the reduced row echelon form of A?

1a) $\vec{x}=\vec{0}$ is a solution to $A\vec{x}=\vec{0}$, no matter what A is $(A\vec{0}=\vec{0})$

in this case, backsolve

$x_1 = -2x_3 - x_4 - 3x_5$
$x_2 = 2x_3 + 3x_4 - x_5$
$x_3 = $ free
$x_4 = $ free
$x_5 = $ free

$$\begin{bmatrix} x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix} = x_3\begin{bmatrix}-2\\2\\1\\0\\0\end{bmatrix} + x_4\begin{bmatrix}-1\\3\\0\\1\\0\end{bmatrix} + x_5\begin{bmatrix}-3\\-1\\0\\0\\1\end{bmatrix}$$

where $x_3, x_4, x_5 \in \mathbb{R}$ are arbitrary

1b) soltns almost the same.

$x_1 = c_1 - 2x_3 - x_4 - 3x_5$
$x_2 = c_2 + 2x_3 + 3x_4 - x_5$
$x_3 = $ free
$x_4 = $ free
$x_5 = $ free

$$\begin{bmatrix} x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix} = \begin{bmatrix}c_1\\c_2\\0\\0\\0\end{bmatrix} + \text{homog solns}$$

1b)
1c) ∞'ly many soltns. # free params = # non-pivot columns in coefficient matrix

$= 3.$

Exercise 2) Now consider the matrix B and similar questions: B is "3×2" matrix

$$B := \begin{bmatrix} 1 & 2 \\ -1 & 3 \\ 4 & 2 \end{bmatrix} \begin{matrix} 0 & b_1 \\ 0 & b_2 \\ 0 & b_3 \end{matrix} \qquad rref(B) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{matrix} 0 & c_1 \\ 0 & c_2 \\ 0 & c_3 \end{matrix}$$

2a) How many solutions to the homogeneous problem $B\underline{\mathbf{x}} = \underline{\mathbf{0}}$? $\vec{x} = \vec{0}$ $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

backsolve aug. matrix: $x_1 = 0$, $x_2 = 0$

ONE

2b) Is the inhomogeneous problem $B\underline{\mathbf{x}} = \underline{\boldsymbol{b}}$ solvable for every right side vector $\underline{\boldsymbol{b}}$?

NO only solvable if $c_3 = 0$.

2c) When the inhomogeneous problem is solvable, how many solutions does it have?

ONE : $x_1 = c_1$, $x_2 = c_2$

<u>Exercise 3)</u> Square matrices (i.e number of rows equals number of columns) with 1's down the diagonal which runs from the upper left to lower right corner are special. They are called <u>identity matrices,</u> I (because $I\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{x}}$ is always true (as long as the vector $\underline{\boldsymbol{x}}$ is the right size)).

C is 4×4

$$C := \left[\begin{array}{cccc|c} 1 & 0 & -1 & 1 & 0 \quad b_1 \\ 22 & -1 & 3 & 5 & 0 \quad b_2 \\ 7 & 4 & 6 & 2 & 0 \quad b_3 \\ 3 & 5 & 7 & 13 & 0 \quad b_4 \end{array}\right] \qquad rref(C) = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \quad c_1 \\ 0 & 1 & 0 & 0 & 0 \quad c_2 \\ 0 & 0 & 1 & 0 & 0 \quad c_3 \\ 0 & 0 & 0 & 1 & 0 \quad c_4 \end{array}\right]$$

<u>3a)</u> How many solutions to the homogeneous problem $C\underline{\mathbf{x}} = \underline{\mathbf{0}}$?

ONE $\quad \vec{x} = \vec{0}$

<u>3b)</u> Is the inhomogeneous problem $C\underline{\mathbf{x}} = \underline{\boldsymbol{b}}$ solvable for every choice of $\underline{\boldsymbol{b}}$?

YES each row of rref(c) has a pivot

<u>3c)</u> How many solutions?

ONE

" $\vec{x} = \vec{c}$ "

why is I called identity matrix?

Ans

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} := x_1\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$I\vec{x} = \vec{x}$.

<u>Exercise 4:</u> What are your general conclusions?

4a) What conditions on the reduced row echelon form of the matrix A guarantee that the homogeneous equation $A\underline{\mathbf{x}} = \underline{\mathbf{0}}$ has infinitely many solutions? a positive # of "free variable" i.e. there are non-pivot columns for A

4b) What condition on the reduced row echelon form of A guarantees that the nonhomogeneous equation $A\underline{\mathbf{x}} = \underline{\boldsymbol{b}}$ always has at least one solution (i.e. is consistent), no matter what the entries of $\underline{\boldsymbol{b}}$ are?

rref (A) has no zero rows, i.e. every row has a pivot.

4c) What conditions on the numbers of rows and columns of A always force infinitely many solutions to the homogeneous problem?

$A\vec{x} = \vec{0}$ If more columns than rows, then there will be non-pivot columns

4d) What conditions on the numbers of rows and columns of A guarantee that there will be lots of vectors $\underline{\boldsymbol{b}}$ for which $A\underline{\mathbf{x}} = \underline{\boldsymbol{b}}$ is inconsistent? If more rows than columns then rref(A) has zero rows

4e) What conditions on the reduced row echelon form of A guarantee that solutions $\underline{\mathbf{x}}$ to $A\underline{\mathbf{x}} = \underline{\boldsymbol{b}}$ are always unique (if they exist)?

every column is a pivot column (no free variables)
ex. #2

4f) If A is a square matrix (m=n), what can you say about the solution set to $\mathbf{A}\underline{\mathbf{x}} = \underline{\boldsymbol{b}}$ when

- The reduced row echelon form of A is the identity matrix? $\longrightarrow$ exactly one solution
- The reduced row echelon form of A is not the identity matrix?

means less than n (=m) pivots
$\Rightarrow$ not every syst is consistent
& when consistent, ∞'ly many soltns.

Math 2270-002
Friday August 31 supplement.

Section 1.1-1.5 textbook Theorems:

Theorem 1 (p. 13 Uniqueness of reduced (row) echelon form) Each matrix is row equivalent to one and only one reduced row echelon form matrix.

Theorem 2 (p. 21 Existence and Uniqueness Theorem) A linear system is consistent if and only if the rightmost column of the augmented matrix is not a pivot column. In other words, no echelon form of the augmented matrix has a row which is all zeroes, except for a non-zero final entry. If the system is consistent then the solution is unique if and only if there are no free variables. (And, the number of free variables equals the number of non-pivot columns in the coefficient matrix.)

Theorem 3 (p. 36 Equivalent formulations for linear systems of equations). Let A be an $m \times n$ matrix, with columns $\underline{\boldsymbol{a}}_1, \underline{\boldsymbol{a}}_2, \ldots \underline{\boldsymbol{a}}_n$ and let $\underline{\boldsymbol{b}}$ be in $\mathbb{R}^m$. Then the matrix equation

$$(1) \qquad A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}$$

is equivalent to the vector equation

$$(2) \qquad x_1\underline{\boldsymbol{a}}_1 + x_2\,\underline{\boldsymbol{a}}_2 + \ldots x_n\,\underline{\boldsymbol{a}}_n = \underline{\boldsymbol{b}}$$

as well as to the linear system of m equations in n unknowns which has augmented matrix with columns

$$(3) \qquad \left[\underline{\boldsymbol{a}}_1, \underline{\boldsymbol{a}}_2, \ldots \underline{\boldsymbol{a}}_n \ \ \underline{\boldsymbol{b}}\ \right].$$

In particular, the solution sets for (1), (2), (3) are all the same.

Theorem 4 (p. 37) Let A be an $m \times n$ matrix. Then the following statements are logically equivalent. That is, for a particular A, either they are all true statements or they are all false:

a) For each $\underline{\boldsymbol{b}}$ in $\mathbb{R}^m$, the equation $A\,\underline{\mathbf{x}} = \underline{\boldsymbol{b}}$ has a solution.

b) Each $\underline{\boldsymbol{b}}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.

c) The columns of A span $\mathbb{R}^m$.

d) A has a pivot position in each row.

Theorem 5 (p. 39 Matrix multiplication is linear) If A is an $m \times n$ matrix, $\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}} \in \mathbb{R}^n$, c a scalar, then

a) $A(\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}}) = A\,\underline{\boldsymbol{u}} + A\,\underline{\boldsymbol{v}}$

b) $A(c\,\underline{\boldsymbol{u}}) = c\,A\,\underline{\boldsymbol{u}}$

Theorem 6 (p. 47 how the solutions to homogeneous and nonhomogeneous problems are related) Suppose the equation $A\,\underline{\mathbf{x}} = \underline{\boldsymbol{b}}$ is consistent for some given $\underline{\boldsymbol{b}}$, and let $\underline{\boldsymbol{p}}$ be a solution. Then the solution set of $A\,\underline{\mathbf{x}} = \underline{\boldsymbol{b}}$ is the set of all vectors of the form $\underline{\boldsymbol{w}} = \underline{\boldsymbol{p}} + \underline{\boldsymbol{v}}_h$ where $\underline{\boldsymbol{v}}_h$ is any solution of the homogeneous equation $A\,\underline{\mathbf{x}} = \underline{\mathbf{0}}$.

Fri ~~Sept~~ August 31
- 1.6 some applications

Announcements: 2nd HW is posted (1.4-1.6)
- HW1 sols, Quiz2 sols also } CANVAS.
- these applications are quick (in notes), first finish Wed notes

Warm-up Exercise: Exercise 1 in Wednesday notes

.

1.6 Some applications of matrix equations.

Exercise 1) Balance the following chemical reaction equation, for the burning of propane:

$$x_1\, C_3H_8 + x_2\, O_2 \ \rightarrow x_3\, CO_2 + x_4\, H_2O$$

$$\begin{bmatrix} \#C \\ \#H \\ \#O \end{bmatrix} : \quad x_1 \begin{bmatrix} 3 \\ 8 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

$$x_1 \begin{bmatrix} 3 \\ 8 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} - x_3 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - x_4 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{matrix} 3 & 0 & -1 & 0 \\ 8 & 0 & 0 & -2 \\ 0 & 2 & -2 & -1 \end{matrix}\right| \begin{matrix} 0 \\ 0 \\ 0 \end{matrix}$$

$$x_1\, C_3H_8 + x_2\, O_2 \longrightarrow x_3\, CO_2 + x_4\, H_2O$$

Hint: after you set up the problem, the following reduced row echelon form computation will be helpful:

$$\left[\begin{array}{cccc|c} 3 & 0 & -1 & 0 & 0 \\ 8 & 0 & 0 & -2 & 0 \\ 0 & 2 & -2 & -1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & -\frac{1}{4} & 0 \\ 0 & 1 & 0 & -\frac{5}{4} & 0 \\ 0 & 0 & 1 & -\frac{3}{4} & 0 \end{array}\right]$$

$x_1 = \frac{1}{4} x_4$

$x_2 = \frac{5}{4} x_4$

$x_3 = \frac{3}{4} x_4$

$x_4 = \text{free}$

let $x_4 = 4$

$x_1 = 1$

$x_2 = 5$

$x_3 = 3$

$x_4 = 4$

$1 \cdot C_3H_8 + 5O_2 \rightarrow 3CO_2 + 4H_2O$

Exercise 2) Consider the following traffic flow problem (from our text): What are the possible flow patterns, based on the given information and that the streets are one-way, so none of the flow numbers can be negative?

EXAMPLE 2 The network in Figure 2 shows the traffic flow (in vehicles per hour) over several one-way streets in downtown Baltimore during a typical early afternoon. Determine the general flow pattern for the network.

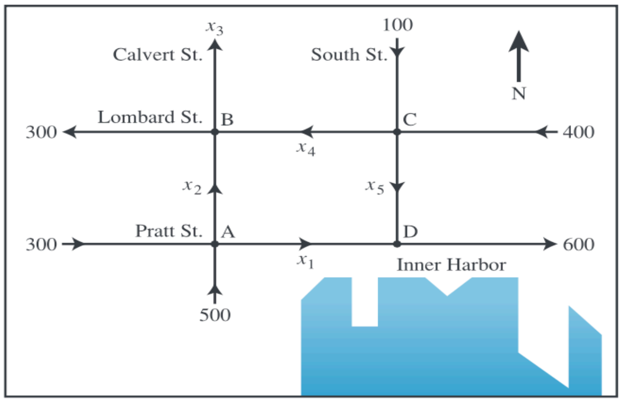


FIGURE 2 Baltimore streets.

Hint: If you set up the flow equations for intersections *A*, *B*, *C*, D in that order, the following reduced row echelon form computation may be helpful:

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 800 \\ 0 & 1 & -1 & 1 & 0 & 300 \\ 0 & 0 & 0 & 1 & 1 & 500 \\ 1 & 0 & 0 & 0 & 1 & 600 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 600 \\ 0 & 1 & 0 & 0 & -1 & 200 \\ 0 & 0 & 1 & 0 & 0 & 400 \\ 0 & 0 & 0 & 1 & 1 & 500 \end{bmatrix}$$

$0 \leq x_5 \leq 500$