Syllabus for Math 2270-002 Linear Algebra

Fall 2018

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office hours M 2:00-3:00 p.m., T 10:00-11:00 a.m, and by appointment in LCB 204. Also available after class (briefly). (these may change)

Lecture MTWF 12:55-1:45 p.m. MWF in ST 208, T in LCB 225

Course websites

Daily lecture notes and weekly homework assignments will be posted on our public home page.

http://www.math.utah.edu/∼korevaar/2270fall18

There are blank spaces in the notes where we will work out examples and fill in details together. Research has shown that class attendance with active participation - including individual and collaborative problem solving, and writing notes by hand - are effective ways to learn class material for almost everyone. Passively watching a lecture is not usually effective. Class notes will be posted at least several days before we use them, and I plan to bring weekly packets to class for you to use. Beyond what's outlined in the notes, there will often be additional class discussion related to homework and other problems.

Grades and exam material will be posted on our CANVAS course page; access via Campus Information Systems.

Textbook *Linear Algebra and its Applications*, 5th edition, by David D. Lay. ISBN: 032198238X

Final Exam logistics: Wednesday, December 12, 1:00-3:00 p.m., in our MWF classroom WEB ST 208. This is the University scheduled time and location.

Catalog description for Math 2270: Euclidean space, linear systems, Gaussian elimination, determinants, inverses, vector spaces, linear transformations, quadratic forms, least squares and linear programming, eigenvalues and eigenvectors, diagonalization. Includes theoretical and computer lab components.

wikipedia Linear Algebra, it's pretty good.

Course Overview: This course is the first semester in a year long sequence (2270-2280) devoted to linear mathematics. In this course, we study two objects: vectors and matrices. We start by thinking of vectors and matrices as arrays of numbers, then we progress to thinking of vectors as elements of a vector space and matrices as representing linear transformations. In our study of vectors and matrices, we learn to solve systems of linear equations, familiarize ourselves with matrix algebra, and explore the theory of vector spaces. Some key concepts we study are determinants, eigenvalues and eigenvectors, orthogonality, symmetric matrices, and quadratic forms. Along the way we encounter applications of the material in math, computer science, statistics and elsewhere. Students who continue into Math 2280 will see that linear algebra is one of the foundations, together with Calculus, upon which the study of differential equations is based.

Prerequisites: C or better in MATH 2210 or MATH 1260 or MATH 1321 or MATH 1320. Practically speaking, you are better prepared for this course if your grades in the prerequisite courses were above the "C" level.

Students with disabilities: The University of Utah seeks to provide equal access to its programs, services and activities for people with disabilities. If you will need accommodations in the class, reasonable prior notice needs to be given to the Center for Disability Services, 162 Olpin Union Building, 581-5020. CDS will work with you and the instructor to make arrangements for accommodations. All information in this course can be made available in alternative format with prior notification to the Center for Disability Services.

Grading

Math 2270-002 is graded on a curve. By this I mean that the final grading scale may end up lower than the usual 90/80/70% cut-offs. **note:** In order to receive a grade of at least "C" in the course you must earn a grade of at least "C" on the final exam. Typical grade distributions in Math 2270 end up with grades divided roughly in thirds between A's, B's, and the remaining grades. Individual classes vary.

Details about the content of each assignment type, and how much they count towards your final grade are as follows:

- Homework (30%): There will be one homework assignment each week. Homework problems will be posted on our public page, and homework assignments will be due in class on Wednesdays. Homework assignments must be stapled. Unstapled assignments will not receive credit. I understand that sometimes homework cannot be completed on time due to circumstances beyond your control. To account for this, each student will be allowed to turn in **two** late homework assignments throughout the course of the semester. These assignments cannot be turned in more than one week late, and must be turned in on a Wednesday with the next homework assignment. You do not need to tell me the reason why your homework assignment is late. Homework will be a mixture of problems from the text and custom problems, and will vary from computational practice to conceptual questions, and will include applications that may sometimes require technology to complete.

- Quizzes (10%): At the end of most Wednesday classes, a short 1-2 problem quiz will be given, taking roughly 10 minutes to do. The quiz will cover relevant topics from the week's lectures, homework, and food for thought work. Two of a student's lowest quiz scores will be dropped. There are no makeup quizzes. You will be allowed and encouraged to work together on these quizzes.

- Midterm exams (30%): Two class-length midterm exams will be given, On Friday September 28 and Friday November 9. I will schedule a room for review on the Thursday before each midterm, at our regular class time of 12:55-1:45 p.m. No midterm scores are dropped.

- Final exam (30%): A two-hour comprehensive exam will be given at the end of the semester. As with the midterms, a practice final will be posted. Please check the final exam time, which is the official University scheduled time. It is your responsibility to make yourself available for that time, so make any arrangements (e.g., with your employer) as early as possible.

Strategies for success

- Attend class regularly, and participate actively.
- Read or skim the relevant text book sections and lecture note outlines *before* you attend class.
- Ask questions and become involved.
- Plan to do homework daily; try homework on the same day that the material is covered in lecture; do not wait until just before homework is due to begin serious work.
- Form study groups with other students.

Learning Objectives for 2270

Computation vs. Theory: This course is a combination of computational mathematics and theoretical mathematics. By theoretical mathematics, I mean abstract definitions and theorems, instead of calculations. The computational aspects of the course may feel more familiar and easier to grasp, but I urge you to focus on the theoretical aspects of the subject. Linear algebra is a tool that is heavily used in mathematics, engineering, science and computer science, so it will likely be relevant to you later in your career. When this time comes, you will find that the computations of linear algebra can easily be done by computing systems such as Matlab, Maple, Mathematica or Wolfram alpha. But to understand the significance of these computations, a person must understand the theory of linear algebra. Understanding abstract mathematics is something that comes with practice, and takes more time than repeating a calculation. When you encounter an abstract concept in lecture and the text, I encourage you to pause and give yourself some time to think about it. Try to give examples of the concept, and think about what the concept is good for.

The essential topics

Be able to find the solution set to linear systems of equations systematically, using row reduction techniques and reduced row echelon form - by hand for smaller systems and using technology for larger ones. Be able to solve (linear combination) vector equations using the same methods, as both concepts are united by the common matrix equation $A\mathbf{x} = \mathbf{b}$.

Be able to use the correspondence between matrices and linear transformations - first for transformations between $\mathbb{R}^n$ and $\mathbb{R}^m$, and later for transformations between arbitrary vector spaces.

Become fluent in matrix algebra techniques built out of matrix addition and multiplication, in order to solve matrix equations.

Understand the algebra and geometry of determinants so that you can compute determinants, with applications to matrix inverses and to oriented volume expansion factors for linear transformations.

Become fluent in the language and concepts related to general vector spaces: linear independence, span, basis, dimension, and rank for linear transformations. Understand how change of basis in the domain and range effect the matrix of a linear transformation.

Be able to find eigenvalues and eigenvectors for square matrices. Apply these matrix algebra concepts and matrix diagonalization to understand the geometry of linear transformations and certain discrete dynamical systems.

Understand how orthogonality and angles in $\mathbf{R}^2, \mathbf{R}^3$ generalize via the dot product to $\mathbf{R}^n$, and via general inner products to other vector spaces. Be able to use orthogonal projections and the Gram-Schmidt process, with applications to least squares problems and to function vector spaces.

Know the spectral theorem for symmetric matrices and be able to find their diagonalizations. Relate this to quadratic forms, constrained optimization problems, and to the singular value decomposition for matrices. Learn some applications to image processing and/or statistics.

Week-by-Week Topics Plan

Topic schedule is subject to slight modifications as the course progresses, but exam dates are fixed.

Week 1: 1.1-1.3; systems of linear equations, row reduction and echelon forms, vector equations.

Week 2: 1.3-1.5; matrix equations, solution sets of linear systems, applications.

Week 3: 1.6-1.8; applications, linear dependence and independence, linear transformations and matrices.

Week 4: 1.8-1.9, 2.1-2.2; linear transformations of the plane, introduction to matrix algebra.

Week 5: 2.3-2.5; matrix inverses, partitioned matrices and matrix factorizations.

Week 6: 3.1-3.3; determinants, algebraic and geometric properties and interpretations. **Midterm exam 1 on Friday September 28** covering material from weeks 1-6.

Week 7: 4.1-4.3; vector spaces and subspaces, nullspaces and column spaces, general linear transformations.

Week 8: 4.3-4.6; linearly independent sets, bases, coordinate systems, dimension and rank

Week 9: 4.6-4.7, 5.1-5.2; change of basis, eigenvectors and eigenvalues, and how to find them.

Week 10: 5.3-5.5; diagonalization, eigenvectors and linear transformations, complex eigendata.

Week 11: 5.6, 6.1-6.2; discrete dynamical systems, introduction to orthogonality. **Midterm exam 2 on Friday November 9** covering material from weeks 7-11

Week 12: 6.3-6.5; orthogonal projections, Gram-Schmidt process, least squares solutions

Week 13: 6.6-6.8; applications to linear models; inner product spaces and applications with Fourier series.

Week 14: 7.1-7.3; diagonalization of symmetric matrices, quadratic forms, constrained and unconstrained optimization.

Week 15: 7.4; singular-value decomposition, applications, course review.

Week16: Final exam Wednesday December 12, 1:00 - 3:00 p.m. in classroom WEB ST 208. This is the University scheduled time.

Math 2270-002 Week 1 notes

We will not necessarily finish the material from a given day's notes on that day. Or on an amazing day we may get farther than I've predicted. We may also add or subtract some material as the week progresses, but these notes represent an outline of what we will cover. These week 1 notes are for sections 1.1-1.3.

Monday August 20:

- Course Introduction
- 1.1: Systems of linear equations

• Pick up syllabus & notes
• write name & what you hope to get out of 2270 (hopes & fears); & a bit about you
hand in at end of class

• Go over course information on syllabus and course homepage:

- http://www.math.utah.edu/~korevaar/2270fall18

• Note that there is a quiz this Wednesday on section 1.1-1.2 material. Your first homework assignment will be due next Wednesday, August 29.

Then, let's begin!

• What is a linear equation?

any equation in some number of variables $x_1, x_2, \ldots x_n$ which can be written in the form

$$a_1 x_1 + a_2 x_2 + \ldots + a_n x_n = b$$

where b and the coefficients $a_1, a_2, \ldots a_n$ are real or complex numbers usually known in advance.

Question: Why do you think we call an equation like that "linear"?

• in 2 variables the sol'n set to
$ax + by = c$
is a line
• all variables are to the 1st power (no higher order polys or weird compositions).

Exercise 1) Which of these is a linear equation?

1a) For the variables x, y

$$3x + 4y = 6$$

yes

1b) For the variables s, t

$$2t = 5 - \sqrt{3}\, s$$

yes $2t + \sqrt{3}\, s = 5$

1c) For the variables x, y

$$2x = 5 - 3\underbrace{\sqrt{y}}_{}\,.$$

no

$\sqrt{y}$ bad.

- What is a system of linear equations?

A general linear system (LS) of m equations in the n variables $x_1, x_2, \dots x_n$ is a list of $m \geq 1$ equations that can be written as

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad\qquad & \;\;\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned}$$

i: which eqtn j: which variable a_{ij} is coeff of.

In such a linear system the coefficients a_{ij} and the right-side number b_j are usually known. The goal is to find all possible values for the *vector* $\underline{\mathbf{x}} = [x_1, x_2, \dots x_n]$ of variables, so that all equations are true. (Thus this is often called finding "simultaneous" solutions to the linear system, because all equations will be true at once.)

Definition *The solution set of a system of linear equations is the collection of all solution vectors to that system.*

Notice that we use two subscripts for the coefficients a_{ij} and that the first one indicates which equation it appears in, and the second one indicates which variable its multiplying; in the corresponding *coefficient matrix A*, this numbering corresponds to the row and column of a_{ij}:

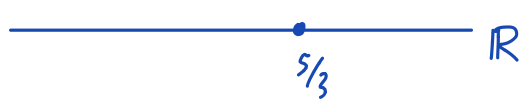
$$A := \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}$$

Let's start small, where geometric reasoning will help us understand what's going on when we look for solutions to linear equations and to linear systems of equations.

Exercise 2: Describe the solution set of each single linear equation below; describe and sketch its geometric realization in the indicated Euclidean space.

2a) $3x = 5$, for $x \in \mathbb{R}$.

$x = 5/3$



$\{5/3\}$. single point

2b) $2x + 3y = 6$, for $[x, y] \in \mathbb{R}^2$.

solution set is a line. L

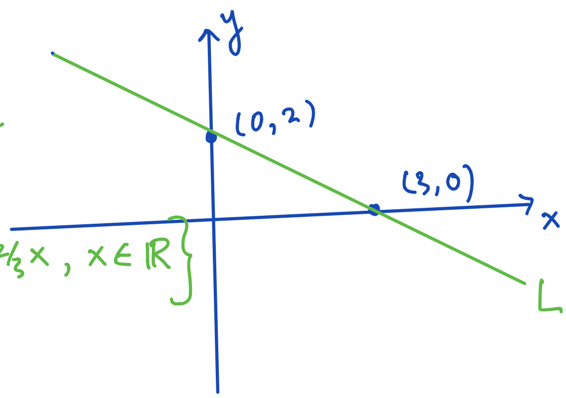
$3y = 6 - 2x$

$y = 2 - 2/3\,x$

$\{(x,y) \in \mathbb{R}^2 \text{ s.t. } y = 2 - 2/3\,x,\ x \in \mathbb{R}\}$

" is an element of

such that



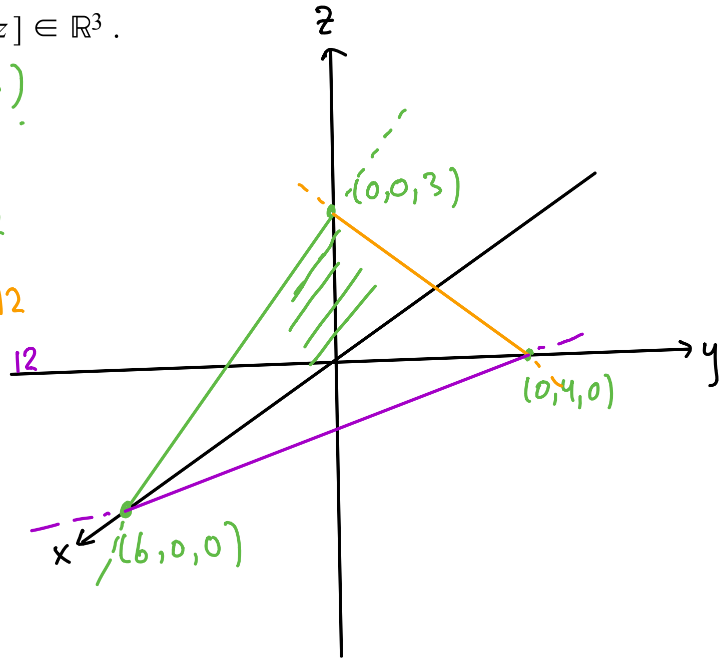
2c) $2x + 3y + 4z = 12$, for $[x, y, z] \in \mathbb{R}^3$.

plane (in Calc 3)

$y = 0: 2x + 4z = 12$

$x = 0: 3y + 4z = 12$

$z = 0: 2x + 3y = 12$



solve for

$z = 3 - \frac{1}{2}x - \frac{3}{4}y$

this also shows it's a plane.

2 linear equations in 2 unknowns:

$$a_{11}\,x + a_{12}\,y = b_1$$
$$a_{21}\,x + a_{22}\,y = b_2$$

goal: find all $[x, y]$ making both of these equations true at once. Since the solution set to each single equation is a line one geometric interpretation is that you are looking for the intersection of two lines.

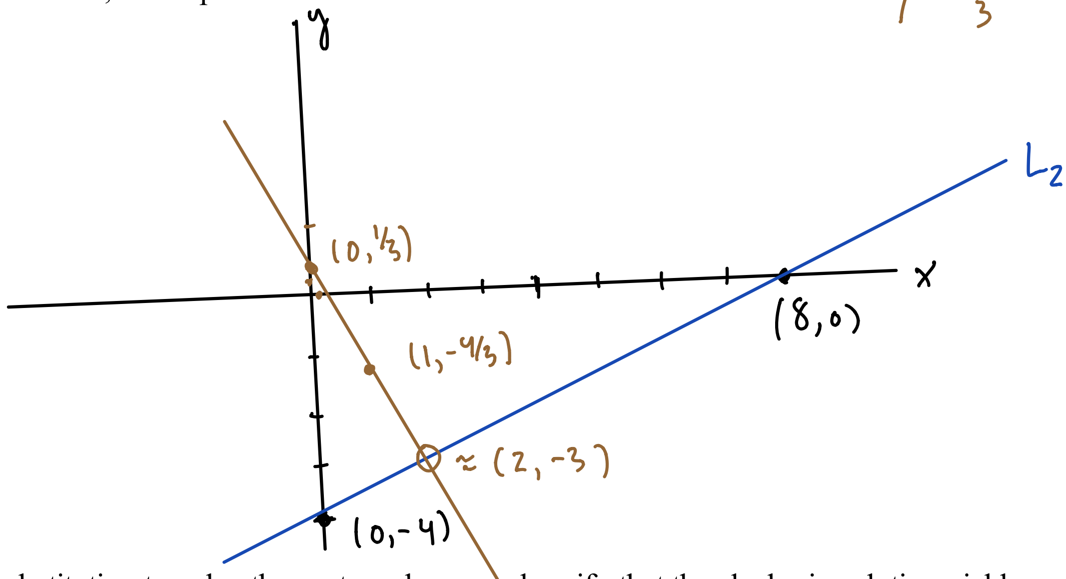
Exercise 3: Consider the system of two equations E_1, E_2:

$$E_1 \qquad 5\,x + 3\,y = 1$$
$$E_2 \qquad x - 2\,y = 8$$

$3y = 1 - 5x \qquad y = \frac{1}{3} - \frac{5}{3}x$

$x = 1$
$y = -\frac{4}{3}$

3a) Sketch the solution set in $\mathbb{R}^2$, as the point of intersection between two lines.



3b) Use the method of substitution to solve the system above, and verify that the algebraic solution yields the point you estimated geometrically in part a.

works great for all sorts of systems, as long as only 2 variables

$E_2 \quad x = 8 + 2y$

$E_1 \to \quad 5(8 + 2y) + 3y = 1$

$13y + 40 = 1$

$13y = -39, \Rightarrow y = -3; \quad x = 8 - 6 = 2.$

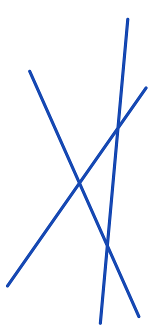
$x = 2$
$y = -3$

with more than 2 variables substitution gets messy fast

3c) What are the possible geometric solution sets to 1, 2, 3, 4 or any number of linear equations in two unknowns?

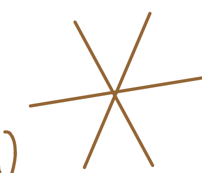
- ∞'ly many solutions: each eqtn had same line as its solution set

  e.g $\begin{cases} 5x + 3y = 1 \\ 10x + 6y = 2 \end{cases}$



- 1 solution: there is one point (& no more) on all of the lines

OR!



- no solutions e.g. 2 parallel lines (or at least 2 of the lines are parallel)

Math 2270-002
Tues Aug 21
- 1.1 Systems of linear equations and Gaussian elimination
- 1.2 Row reduction, echelon form, and reduced row echelon form

Announcements:
- Our Canvas page is live.
- I've posted HW #1 on public page & on CANVAS ← Start HW soon!
- Chapter 1 of text is on CANVAS
- Quiz 1 tomorrow ☺ on Gaussian elimination

'til 12:58

Warm-up Exercise: a) Sketch the two lines

$$\begin{cases} 5x + 3y = 1 \\ \quad x - 2y = 8 \end{cases}$$

and estimate their point of intersection from your sketch.

b) Solve the system in (a) algebraically. Compare to sketch

(this is Exercise 3 in yesterday's notes ~ so you can do your work there.)
3ab

(b) solution $(x,y) = (2,-3)$ writing as a point

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$ writing as position vectors

This is the system we solved at the end of class on Monday:

$$\begin{cases} E_1 & 5\,x + 3\,y = 1 \\ E_2 & x - 2\,y = 8 \end{cases}$$

The method of substituion that we used then does not work well for systems of equations with lots of variables and lots of equations. What does work well is *Gaussian elimination*, which we now explain and illustrate.

great for linear systems with more than two variables

1a) Use the following three "elementary equation operations" to systematically reduce the system $\{E_1, E_2\}$ to an equivalent system (i.e. one that has the same solution set), but of the form

$$1\,x + 0\,y = c_1$$
$$0\,x + 1\,y = c_2$$

(so that the solution is $x = c_1, y = c_2$). Make sketches of the intersecting lines, at each stage.

The three types of elementary equation operation that we use are listed below. Can you explain why the solution set to the modified system is the same as the solution set before you make the modification?

- interchange the order of the equations
- multiply one of the equations by a non-zero constant
- replace an equation with its sum with a multiple of a different equation.

} won't change soln set

does not effect whether all of the eqtns are true.

if $a = a$ then $ca = ca$. If $ca = ca$ & $c \neq 0 \Rightarrow a = a$ (divide by c)

if $\begin{matrix} a = a \\ b = b \end{matrix}$ then $\begin{matrix} a = a \\ cb = cb \end{matrix}$ then $a + cb = a + cb$ is true.

reversible: if $\begin{cases} a + cb = a + cb \\ b = b \end{cases} \Rightarrow \begin{array}{r} a + cb = a + cb \\ -\ \ cb = cb \\ \hline a = a. \end{array}$

$E_1 \quad 5x + 3y = 1$
$E_2 \quad x - 2y = 8$

$E_2 \to E_1 \quad 1 \cdot x - 2y = 8$
$E_1 \to E_2 \quad 5x + 3y = 1$

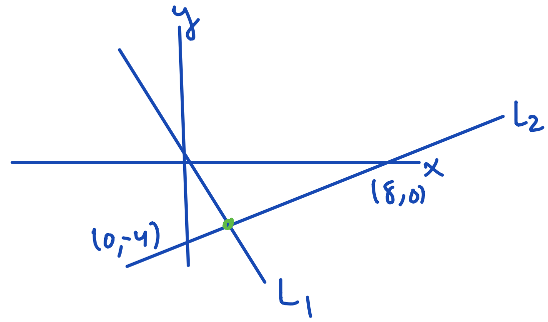
changed names of lines

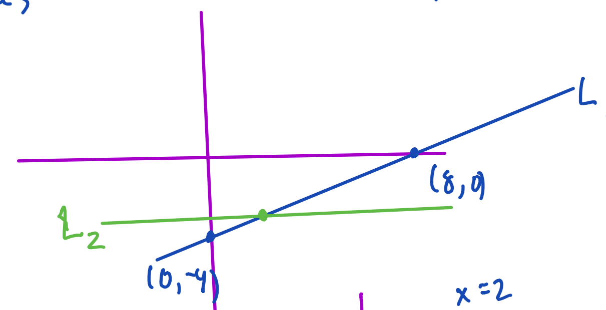
$x - 2y = 8$
$-5E_1 + E_2 \to E_2 \quad 13y = -39$

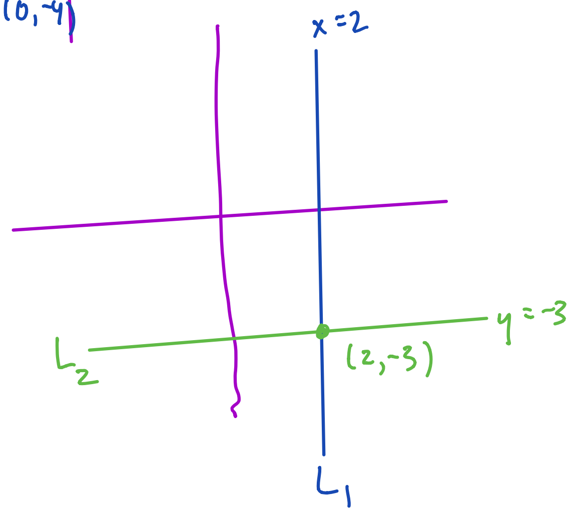
$x - 2y = 8$
$\frac{1}{13}E_2 \to E_2 \quad 1 \cdot y = -3$

$2E_2 + E_1 \to E_1 \quad x = 2$
$y = -3$

$x = 2$
$y = -3$







1b) Look at our work in 1a. Notice that we could have saved a lot of writing by doing this computation "synthetically", i.e. by just keeping track of the coefficients in front of the variables and right-side values:
We represent the system

$$\begin{array}{ll} E_1 & 5x + 3y = 1 \\ E_2 & x - 2y = 8 \end{array}$$

with the *augmented matrix*

coeff matrix

augment with right side values

$$\begin{array}{c} \\ \left[\begin{array}{cc|c} 5 & 3 & 1 \\ 1 & -2 & 8 \end{array}\right] \\ \begin{array}{cc} x\text{-coefs} & y\text{ coef's} \end{array} \end{array}.$$

Using R_1, R_2 as symbols for the rows, redo the operations from part (a). Notice that when you operate synthetically the "elementary equation operations" correspond to "elementary row operations":

- interchange two rows
- multiply a row by a non-zero number
- replace a row by its sum with a multiple of another row.

$$\left[\begin{array}{cc|c} 5 & 3 & 1 \\ 1 & -2 & 8 \end{array}\right]$$

$R_2 \to R_1$, $R_1 \to R_2$

$$\left[\begin{array}{cc|c} 1 & -2 & 8 \\ 5 & 3 & 1 \end{array}\right]$$

$-5R_1 + R_2 \to R_2$

$$\left[\begin{array}{cc|c} 1 & -2 & 8 \\ 0 & 13 & -39 \end{array}\right]$$

$\frac{R_2}{13} \to R_2$

$$\left[\begin{array}{cc|c} 1 & -2 & 8 \\ 0 & 1 & -3 \end{array}\right]$$

$2R_2 + R_1 \to R_1$

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -3 \end{array}\right]$$

uncloak

$1x + 0y = 2$
$0x + 1y = -3$

$x = 2$
$y = -3$

Math 2270-002
Wed Aug 22

- 1.2 Gaussian elimination, row echelon form, and reduced row echelon form.
- Quiz today at end of class, on section 1.1-1.2 material

Announcements:
- quiz today
- from now on, we meet here MTWF ST 208

update: on Thursday the University moved our classroom for accessibility reasons. New pattern:
MWF 12:55-1:45 LCB 215
T 12:55-1:45 LCB 225

Warm-up Exercise: For a linear system of equations in 3 unknowns (say x, y, z or x_1, x_2, x_3), what are the possible geometric solution sets in $\mathbb{R}^3$? Hint: think of ways in which planes can intersect.

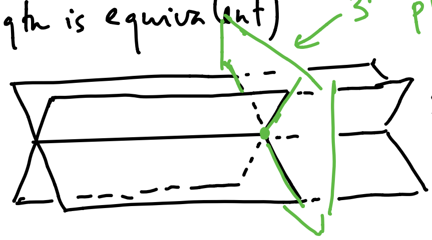
- solution set for each single eqtn is a plane in $\mathbb{R}^3$ (except for the eqtn $0x+0y+0z=0$ which has $\mathbb{R}^3$ as solution set)

possible soltn sets

- $\mathbb{R}^3$ see above.
- plane. (every eqtn is equivalent)

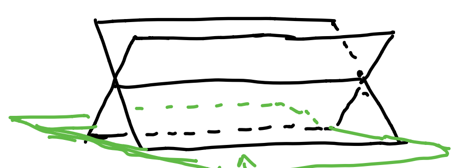
3rd plane

- line e.g.



2 or more planes

- point e.g.
- $\emptyset$ empty set. no solution e.g. 2 parallel planes or lots of other possibilities



3rd plane

Solutions to linear equations in 3 unknowns:

What is the geometric question we're answering in these cases?

Exercise 1) Consider the system

$$x + 2y + z = 4$$
$$3x + 8y + 7z = 20$$
$$2x + 7y + 9z = 23\,.$$

Use elementary equation operations (or if you prefer, elementary row operations in the synthetic version) to find the solution set to this system. The systematic algorithm we use is called Gaussian elimination. (Check Wikipedia about Gauss - he was amazing!)

Hint: The solution set is a single point, $[x, y, z] = [5, -2, 3]$.

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 3 & 8 & 7 & 20 \\ 2 & 7 & 9 & 23 \end{array}\right]$$

$$\begin{array}{r} \\ -3R_1+R_2\to R_2 \\ -2R_1+R_3\to R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 2 & 4 & 8 \\ 0 & 3 & 7 & 15 \end{array}\right]$$

$$\begin{array}{r} \\ \frac{1}{2}R_2\to R_2 \\ \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & 2 & 4 \\ 0 & 3 & 7 & 15 \end{array}\right]$$

$$\begin{array}{r} \\ \\ -3R_2+R_3\to R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right] \longleftarrow \text{row } \cancel{\text{the}} \text{ echelon form.}$$

$$\begin{aligned} x + 2y + z &= 4 \\ y + 2z &= 4 \\ z &= 3 \end{aligned}$$

$y + 6 = 4 \Rightarrow y = -2$

$x - 4 + 3 = 4$

$x = 5$

can backsolve from here

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 3 \end{bmatrix}$$

$$\begin{array}{r} -R_3+R_1\to R_1 \\ -2R_3+R_2\to R_2 \\ \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\begin{array}{r} -2R_2+R_1\to R_1 \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$x = 5$$
$$y = -2$$
$$z = 3$$

Exercise 2) There are other possibilities. In the two systems below we kept all of the coeffients the same as in Exercise 1, except for a_{33} , and we changed the right side in the third equation, for 2a. Work out what happens in each case.

2a)

$$\begin{aligned} x + 2y + z &= 4 \\ 3x + 8y + 7z &= 20 \\ 2x + 7y + 8z &= 20\,. \end{aligned}$$

2b)

$$\begin{aligned} x + 2y + z &= 4 \\ 3x + 8y + 7z &= 20 \\ 2x + 7y + 8z &= 23\,. \end{aligned}$$

2c) What are the possible solution sets (and geometric configurations) for 1, 2, 3, 4,... equations in 3 unknowns?

$$\left[\begin{array}{ccc|cc} 1 & 2 & 1 & 4 & 4 \\ 3 & 8 & 7 & 20 & 20 \\ 2 & 7 & 8 & 20 & 23 \end{array}\right]$$

$$\begin{array}{l} \\ -3R_1+R_2\to R_2 \\ -2R_1+R_3\to R_3 \end{array} \quad \left[\begin{array}{ccc|cc} 1 & 2 & 1 & 4 & 4 \\ 0 & 2 & 4 & 8 & 8 \\ 0 & 3 & 6 & 12 & 15 \end{array}\right]$$

$$\begin{array}{l} R_2/2\to R_2 \\ R_3/3\to R_3 \end{array} \quad \left[\begin{array}{ccc|cc} 1 & 2 & 1 & 4 & 4 \\ 0 & 1 & 2 & 4 & 4 \\ 0 & 1 & 2 & 4 & 5 \end{array}\right]$$

$$\begin{array}{l} \\ \\ -R_2+R_3\to R_3 \end{array} \quad \left[\begin{array}{ccc|cc} 1 & 2 & 1 & 4 & 4 \\ 0 & 1 & 2 & 4 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{l} -4R_3+R_1\to R_1 \\ -4R_3+R_2\to R_2 \\ \end{array} \quad \left[\begin{array}{ccc|cc} 1 & 2 & 1 & 4 & 0 \\ 0 & 1 & 2 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$-2R_2+R_1\to R_1 \quad \left[\begin{array}{ccc|cc} 1 & 0 & -3 & -4 & 0 \\ 0 & 1 & 2 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

system1

$$\begin{aligned} x \qquad -3z &= -4 \\ y + 2z &= 4 \\ 0 &= 0 \end{aligned}$$

$$\boxed{\begin{aligned} x &= -4+3z \\ y &= 4-2z \\ z &= \text{free} \end{aligned}}$$

system 2.

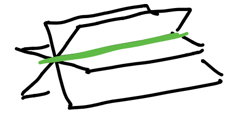
$$\begin{aligned} x \qquad -3z &= 0 \\ y + 2z &= 0 \\ 0x + 0y + 0z &= 1 \end{aligned}$$

no solution

system "inconsistent"

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 3z \\ -2z \\ z \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \\ 0 \end{bmatrix} + z \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

line.
paddle wheels



Summary of the systematic method known as Gaussian elimination for solving systems of linear equations.

We write the linear system (LS) of m equations for the vector $\underline{\boldsymbol{x}} = [x_1, x_2, \ldots x_n]$ of the n unknowns as

$$\begin{aligned} a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n &= b_1 \\ a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n &= b_2 \\ \vdots \qquad\qquad\qquad\qquad \vdots \quad &\;\; \vdots \\ a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n &= b_m \end{aligned}$$

The matrix that we get by adjoining (augmenting) the right-side $\underline{\boldsymbol{b}}$-vector to the coefficient matrix $A = [a_{ij}]$ is called the *augmented matrix* $\langle A | \underline{\boldsymbol{b}} \rangle$:

$$\left[\begin{array}{ccccc|c} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & b_2 \\ \vdots & & & & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} & b_m \end{array}\right]$$

Our goal is to find all the solution vectors $\underline{\boldsymbol{x}}$ to the system - i.e. the *solution set* .

There are three types of *elementary equation operations* that don't change the solution set to the linear system. They are

- interchange two of equations
- multiply one of the equations by a non-zero constant
- replace an equation with its sum with a multiple of a different equation.

And that when working with the augmented matrix $\langle A | \underline{\boldsymbol{b}} \rangle$ these correspond to the three types of *elementary row operations:*

- interchange ("swap") two rows
- multiply one of the rows by a non-zero constant
- replace a row by its sum with a multiple of a different row.

Gaussian elimination: Use elementary row operations and work column by column (from left to right) and row by row (from top to bottom) to first get the augmented matrix for an equivalent system of equations which is in

row-echelon form:

- (1) All "zero" rows (having all entries $= 0$) lie beneath the non-zero rows.
- (2) The leading (first) non-zero entry in each non-zero row lies strictly to the right of the one above it. These entries are called *pivots* in our text, and the corresponding columns are called *pivot columns*.

(At this stage you could "backsolve" to find all solutions.)

$$\begin{array}{ccccc} 2 & * & * & * & * \\ 0 & 0 & 3 & * & * \\ 0 & 0 & 0 & 4 & * \\ 0 & 0 & 0 & 0 & 0 \end{array}$$

Next, continue but by working from bottom to top and from right to left instead, so that you end with an augmented matrix that is in

reduced row echelon form: (1),(2), together with

(3) Each leading non-zero row entry has value 1. Such entries are called "leading $1's$" and are the "pivots" for the corresponding pivot columns.

(4) Each pivot column has $0's$ in all the entries except for the pivot entry of 1.

Finally, read off how to explicitly specify the solution set, by "backsolving" from the reduced row echelon form.

$$\begin{array}{ccccc} 1 & * & 0 & 0 & * \\ 0 & 0 & 1 & 0 & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 \end{array}$$

Note: There are lots of row-echelon forms for a matrix, but only one reduced row-echelon form. (We'll see why later.) All mathematical software packages have a command to find the reduced row echelon form of a matrix.

Math 2270-002
Wed Aug 22

- 1.2 Gaussian elimination, row echelon form, and reduced row echelon form.
- Quiz today at end of class, on section 1.1-1.2 material

Announcements:
- quiz today
- from now on, we meet here MTWF ST 208

update: on Thursday the University moved our classroom for accessibility reasons. New pattern:
MWF 12:55-1:45 LCB 215
T 12:55-1:45 LCB 225

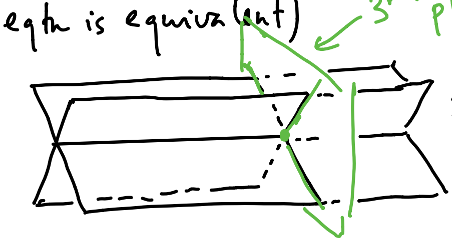
Warm-up Exercise: For a linear system of equations in 3 unknowns (say x, y, z or x_1, x_2, x_3), what are the possible geometric solution sets in $\mathbb{R}^3$? Hint: think of ways in which planes can intersect.

- solution set for each single eqtn is a plane in $\mathbb{R}^3$ (except for the eqtn $0x + 0y + 0z = 0$ which has $\mathbb{R}^3$ as solution set)

possible soltn sets

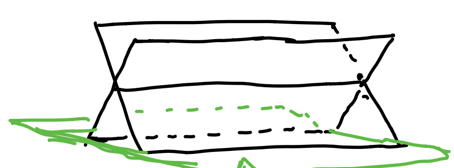
- $\mathbb{R}^3$ see above.
- plane. (every eqtn is equivalent)
- line e.g.

3rd plane



2 or more planes

- point e.g.
- $\emptyset$ empty set. no solution e.g. 2 parallel planes or lots of other possibilities



3rd plane

Review Gaussian elimination, row echelon form and reduced row echelon form from yesterday. Then solve this large system:

Exercise 1 Find all solutions to the system of 3 linear equations in 5 unknowns

$$\begin{aligned} x_1 - 2x_2 + 3x_3 + 2x_4 + x_5 &= 10 \\ 2x_1 - 4x_2 + 8x_3 + 3x_4 + 10x_5 &= 7 \\ 3x_1 - 6x_2 + 10x_3 + 6x_4 + 5x_5 &= 27\,. \end{aligned}$$

Here's the augmented matrix:

$$\left[\begin{array}{rrrrr|r} 1 & -2 & 3 & 2 & 1 & 10 \\ 2 & -4 & 8 & 3 & 10 & 7 \\ 3 & -6 & 10 & 6 & 5 & 27 \end{array}\right]$$

Find the reduced row echelon form of this augmented matrix and then backsolve to explicitly parameterize the solution set. (Hint: it's a two-dimensional plane in $\mathbb{R}^5$, if that helps. :-))

Maple says:

```
> with(LinearAlgebra) :   # matrix and linear algebra library
> A := Matrix(3, 5, [1,-2, 3, 2, 1,
                     2,-4, 8, 3, 10,
                     3,-6, 10, 6, 5]) :
  b := Vector([10, 7, 27]) :
  <A|b>;     # the mathematical augmented matrix doesn't actually have
             # a vertical line between the end of A and the start of b
  ReducedRowEchelonForm(<A|b>);
```

$$\begin{bmatrix} 1 & -2 & 3 & 2 & 1 & 10 \\ 2 & -4 & 8 & 3 & 10 & 7 \\ 3 & -6 & 10 & 6 & 5 & 27 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 0 & 0 & 3 & 5 \\ 0 & 0 & 1 & 0 & 2 & -3 \\ 0 & 0 & 0 & 1 & -4 & 7 \end{bmatrix} \tag{1}$$

```
> LinearSolve(A, b);
      #  this command will actually write down the general solution, using
      # Maple's way of writing free parameters, which actually makes
      # some sense.  Generally when there  are free parameters involved,
      #  there will be equivalent ways to express the solution that may
      # look different.  But usually Maple's version will look like yours,
      # because it's using the same algorithm and choosing the free parameters
      # the same way too.
```

$$\begin{bmatrix} 5 + 2\,\_t_2 - 3\,\_t_5 \\ \_t_2 \\ -3 - 2\,\_t_5 \\ 7 + 4\,\_t_5 \\ \_t_5 \end{bmatrix} \tag{2}$$

```
>
```

Math 2270-002
Fri Aug 24
- 1.3 Vectors and vector equations

Announcements: Start §1.3 .. "vector eqtns" very important "application" that leads to linear systems of eqtns
- review vectors, vector addition, scalar mult.

Review... 'til 12:58 before we discuss

Warm-up Exercise:
- Which of these augmented matrices is in reduced row echelon form?
- Do the associated systems of linear equations have
  (i) no solutions (inconsistent system)
  (ii) a unique solution, i.e. exactly one.
  (iii) ∞'ly many solutions

in rref
1. "zero" rows at bottom
2. pivot locations move to right as you go down rows

(row echelon form)

3. pivots = 1
4. entire pivot col. = 0's except for pivot

RREF

a) $\left[\begin{array}{ccc|c} 1 & 2 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 3 & 15 \end{array}\right]$ NO 3 fails 4 fails (2nd pivot col.) | 1 sol. | row echelon form

b) $\left[\begin{array}{ccc|c} 1 & 0 & 17 & 3 \\ 0 & 1 & 6 & 2 \\ 0 & 0 & 0 & 1 \end{array}\right]$ NO 4 fails in 4th col. | no sol's inconsistent

c) $\left[\begin{array}{cc|c} 0 & 1 & 2 \\ 1 & 0 & 5 \\ 0 & 1 & 2 \end{array}\right]$ NO 2 fails 4 fails | 1 soltn. $\rightarrow$ unknowns x, y eqtns say $y = 2$, $x = 5$, $y = 2$ soltn $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$

rref: $\left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$

d) $\left[\begin{array}{ccc|c} 1 & 0 & 17 & 3 \\ 0 & 1 & 6 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$ YES | ∞'ly many

unknowns x, y, z
$x = 3 - 17t$
$y = 2 - 6t$
$z = \text{free} = "t" \in \mathbb{R}$

1.3 Vectors and vector equations: We'll carefully define vectors, algebraic operations on vectors and geometric interpretations of these operations, in terms of displacements. These ideas will eventually give us another way to interpret systems of linear equations.

Definition: A matrix with only one column, i.e an $n \times 1$ matrix, is a *vector* in $\mathbb{R}^n$. (We also call matrices with only one row "vectors", but in this section our vectors will always be column vectors.)

Examples:

$\underline{\boldsymbol{u}} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ is a vector in $\mathbb{R}^2$. $= \langle 3, -1 \rangle = 3\hat{i} - \hat{j}$

(2210)

$\underline{\boldsymbol{v}} = \begin{bmatrix} 2 \\ -1 \\ 3 \\ 5 \\ 0 \end{bmatrix}$ is a vector in $\mathbb{R}^5$.

- We can multiply vectors by *scalars* (real numbers) and add together vectors that are the same size. We can combine these operations.

Example: For $\underline{\boldsymbol{u}} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ and $\underline{\boldsymbol{w}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ compute

$$\underline{\boldsymbol{u}} + 4\,\underline{\boldsymbol{w}} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} + \begin{bmatrix} 4 \\ 8 \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \end{bmatrix}$$

$$\Rightarrow 4\vec{w} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

$$5\begin{bmatrix} 3 \\ -1 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 13 \\ -9 \end{bmatrix}$$

Definition: For $\underline{\boldsymbol{u}} = \begin{bmatrix} u_1 \\ u_2 \\ : \\ u_n \end{bmatrix}, \underline{\boldsymbol{v}} = \begin{bmatrix} v_1 \\ v_2 \\ : \\ v_n \end{bmatrix} \in \mathbb{R}^n; c \in \mathbb{R}$,

$$\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}} := \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ : \\ u_n + v_n \end{bmatrix} \quad ; \quad c\,\underline{u} := \begin{bmatrix} cu_1 \\ cu_2 \\ : \\ cu_n \end{bmatrix}.$$

$\text{entry}_i\, (\vec{u} + \vec{v}) := u_i + v_i$

$\text{entry}_i\, c\vec{u} = c\, u_i$

Exercise 1 Our vector notation may not be the same as what you used in math 2210 or Math 1320 or 1321. Let's discuss the notation you used, and how it corresponds to what we're doing here.

see above

Vector addition and scalar multiplication have nice algebraic properties:

Let $\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}}, \underline{\boldsymbol{w}} \in \mathbb{R}^n$, $\quad c, d \in \mathbb{R}$. Then

(i) $\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}} = \underline{\boldsymbol{v}} + \underline{\boldsymbol{u}}$

$$\begin{bmatrix}1\\2\\3\end{bmatrix} + \begin{bmatrix}0\\5\\-6\end{bmatrix} = \begin{bmatrix}1\\7\\-3\end{bmatrix} = \begin{bmatrix}0\\5\\-6\end{bmatrix} + \begin{bmatrix}1\\2\\3\end{bmatrix}$$

$\text{entry}_i(\vec{u}+\vec{v}) = u_i + v_i \qquad 1 \le i \le n$

$\text{entry}_i(\vec{v}+\vec{u}) = v_i + u_i$

equal because real # addition is commutative

(ii) $(\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}}) + \underline{\boldsymbol{w}} = \underline{\boldsymbol{u}} + (\underline{\boldsymbol{v}} + \underline{\boldsymbol{w}})$

(iii) $\underline{\boldsymbol{u}} + \underline{\boldsymbol{0}} = \underline{\boldsymbol{0}} + \underline{\boldsymbol{u}} = \underline{\boldsymbol{u}}$

(iv) $\underline{\boldsymbol{u}} + (-\underline{\boldsymbol{u}}) = (-\underline{\boldsymbol{u}}) + \underline{\boldsymbol{u}} = \underline{\boldsymbol{0}}$

(v) $c(\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}}) = c\,\underline{\boldsymbol{u}} + c\,\underline{\boldsymbol{v}}$

the rest of the properties are also true because of real number arithmetic axioms, applied to each entry of the vectors on each side of the identities.

(vi) $(c + d)\underline{\boldsymbol{u}} = c\,\underline{\boldsymbol{u}} + d\,\underline{\boldsymbol{u}}$

(vii) $c(d\,\underline{\boldsymbol{u}}) = (c\,d)\,\underline{\boldsymbol{u}}$

(viii) $1\,\underline{\boldsymbol{u}} = \underline{\boldsymbol{u}}$.

Exercise 2. Verify why these properties hold!

Geometric interpretation of vectors as displacements

The space $\mathbb{R}^n$ may be thought of in two equivalent ways. In both cases, $\mathbb{R}^n$ consists of all possible $n-tuples$ of numbers:

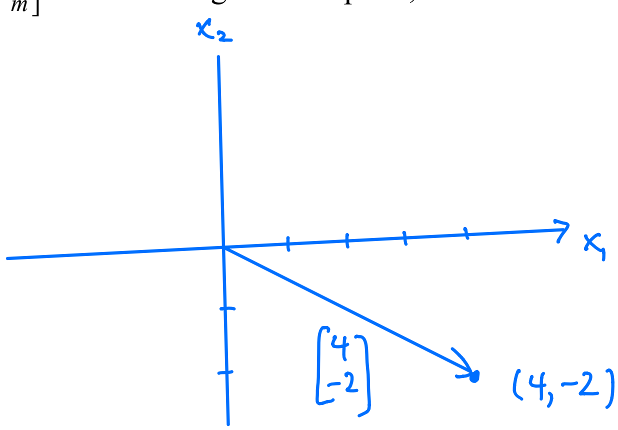
(i) We can think of those $n-tuples$ as representing points, as we're used to doing for $n = 1, 2, 3$. In this case we can write

$$\mathbb{R}^n = \{(x_1, x_2, ..., x_n), s.t.\ x_1, x_2, ..., x_n \in \mathbb{R}\}.$$

(ii) We can think of those $n-tuples$ as representing vectors that we can add and scalar multiply. In this case we can write

$$\mathbb{R}^n = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ : \\ x_n \end{bmatrix}, s.t.\ x_1, x_2, ..., x_n \in \mathbb{R} \right\}.$$

Since algebraic vectors (as above) can be used to measure geometric displacement, one can identify the two models of $\mathbb{R}^n$ as sets by identifying each point $(x_1, x_2, ... x_n)$ in the first model with the displacement vector $\underline{\boldsymbol{x}} = [x_1, x_2, ... x_m]^T$ from the origin to that point, in the second model, i.e. the "position vector" of the point.



Exercise 3) Let $\underline{\boldsymbol{u}} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\underline{\boldsymbol{v}} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

3a) Plot the points $(1, -1)$ and $(1, 3)$, which have position vectors $\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}}$. Draw these position vectors as arrows beginning at the origin and ending at the corresponding points.

3b) Compute $\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}}$ and then plot the point for which this is the position vector. Note that the algebraic operation of vector addition corresponds to the geometric process of composing horizontal and vertical displacements.

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

3c) Compute $\underline{\boldsymbol{u}}$ and $2\,\underline{\boldsymbol{v}}$ and $\underline{\boldsymbol{u}} + 2\,\underline{\boldsymbol{v}}$ plot the corresponding points for which these are the position vectors.

$$2\vec{v} = 2\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix} \qquad \vec{u} + 2\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \underbrace{\begin{bmatrix} 2 \\ 6 \end{bmatrix}}_{2\vec{v}} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

3d) Plot the parametric line segment whose points are the endpoints of the position vectors $\{\underline{\boldsymbol{u}} + t\,\underline{\boldsymbol{v}},\ -1 \le t \le 2\}$.