Math 2270-002 Week 11 notes
We will not necessarily finish the material from a given day's notes on that day. We may also add or subtract some material as the week progresses, but these notes represent an in-depth outline of what we plan to cover. These notes cover material in 5.4-5.5, 6.1.

Mon Nov 5

- 5.4-5.5 Brief intro to Matlab, continued discussion of change of variables to understand linear transformations, and introduction to complex eigenvalues and eigenvectors

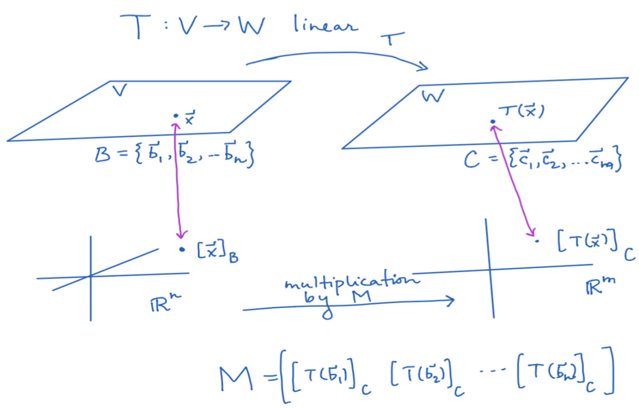
Announcements:

Warm-up Exercise:

Monday Review and look ahead:

We've been studying eigenvectors and eigenvalues for square matrices A, and the notion of diagonalizability, which we first understood in an algebraic sense.

On Friday we talked about how linear transformations $T : V \rightarrow W$ are associated with matrix transformations from $\mathbb{R}^n \rightarrow \mathbb{R}^m$, once we choose bases $B = \{\underline{\boldsymbol{b}}_1, \underline{\boldsymbol{b}}_2, \ldots \underline{\boldsymbol{b}}_n\}$ for V, $C = \{\underline{\boldsymbol{c}}_1, \underline{\boldsymbol{c}}_2, \ldots \underline{\boldsymbol{c}}_m\}$ for W.

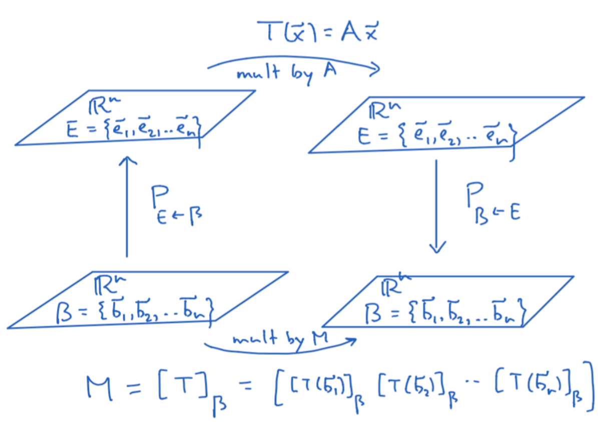


This general framework is connected to matrix diagonalizability, and to more general notion of "similar matrices" when we have linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^n$ expressed in one coordinate system, and wish to change to another one. On Friday we understood how the algebraic identity for diagonalizable matrices

$$D = P^{-1} A P$$

is related to this "change of coordinates" framework: For a diagonalizable matrix A, the identity above is also a change of variables identity for understanding the matrix transformation $T(\underline{\boldsymbol{x}}) = A\,\underline{\boldsymbol{x}}$ first given in standard coordinates; instead with respect to the eigenbasis for $\mathbb{R}^n$, $B = \{\underline{\boldsymbol{b}}_1, \underline{\boldsymbol{b}}_2, \ldots \underline{\boldsymbol{b}}_n\}$which constitutes the columns of P. In fact, the identity above can be read as

$$[T]_\beta = P_{\beta \leftarrow E}\, [T]_E\, P_{E \leftarrow \beta}$$



So far, all of our eigendata has been real, but it's possible to have complex eigenvalues and eigenvectors as well, in which case we switch from considering real number scalars to considering complex number scalars and proceed as before. We'll introduce those ideas today and tomorrow, in section 5.5.

On Wednesday, I tentatively plan to begin Chapter 6 on "Orthogonality". This Chapter will lead to amazing applications, but it begins with a review and extension of dot product ideas related to angles between vectors and orthogonality in $\mathbb{R}^n$, that you learned about in multivariable Calculus for $\mathbb{R}^2$ and $\mathbb{R}^3$. There will be a relatively short homework assignment due next Wednesday, containing preliminary Chapter 6 material.

On Friday we have our second midterm exam, which will cover sections 4.1-4.6, 4.9 (and google page rank), 5.1-5.5. I'll reserve a room for a problem session on Thursday, where we will go over a practice exam.

For your extra credit homework problem, and because you want to familiarize yourself with Matlab for the upcoming chapter as well, I'll demo Matlab.

Here are the two Matlab scripts I've created for this week. You can download these from either our public homework page or CANVAS. Put them into a directory and open them from Matlab. When you run the script with the big arrow at the top of your Matlab window all the commands will be executed. You can modify the script as desired.

The file "matrixcomputations.m" (The .m signifies matlab file.) :

```
% some matlab commands for matrices...if you want more just use google.

% enter a matrix row by row, each row terminates with ";"
A=[1 2 3; 4 5 6]   %use spaces or commas to separate row entries
AA=[3 5 9;2 4 6];  %if you put a semi-colon at the end the computation
                %isn't echoed in the command window.

%rref computes reduced row echelon form.  I named the result
% with an equals sign and a name, so
%that it would be saved to my workspace after running the script

B=rref(A)

% augmenting a matrix, either by rows or columns:  first by row

b=[1,0,0]   %row vector ..
C=[A;b]  % add a 3rd row to A, using b
        % use semicolons to separate rows
D=rref(C) % the 3 by 3 matrix D is invertible, since D is the identity
E=C^(-1)  % inverse matrix...everything is decimals

F= C*E  % this is how you multiply matrices!  should get identity

% augmenting a matrix by columns

g=[-1;3] % a column vector with two entries
H=[A,g] % augment A with the column g
K=rref(H) %reduce the augmented matrix to find solutions to Ax=g

% eigenvalues and eigenvectors

L=[2,1,0;0,2,0;0,0,3]
lambda=eig(L)  % eigenvalues
[M,N]=eig(L)  % eigenvalues in first diagonal matrix,
            %eigenvectors in second matrix
            % note that this matrix is not diagonalizable
```

The file "PAC_team_rankings.m"

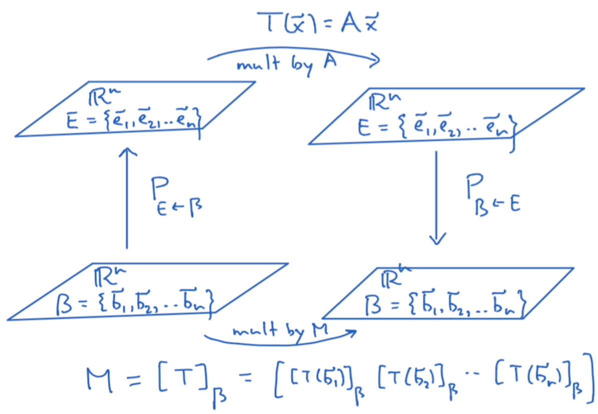
```
% PAC 12 Stochastic Matrix before past weekend
%    AZ  AZS CAL CU  OR  OSU STA UCLA USC UT WA  WAS
SMa=[ 0,  0, 1/3.,0, 1/3,.25, 0,   0,  0,  0, 0,  0;    %AZ
      0,  0,  0,  0,  0, .25, 0,   0  1/3, 0, 0,  0;    %AZS
      0,  0,  0,  0,  0, .25, 0,   0,  0,  0,.5,  0;    %CAL
      0, 1/3 ,0,  0,  0,  0,  0,  1/3, 0,  0, 0,  0;    %CU
      0,  0, 1/3  0,  0,  0,  0,   0,  0,  0,.5,  0;    %OR
      0,  0,  0, 1/3, 0,  0,  0,   0,  0,  0, 0,  0;    %OSU
      0, 1/3, 0,  0, 1/3, 0,  0,   0, 1/3, 0, 0,  0;    %STA
     1/3, 0, 1/3, 0,  0,  0,  0,   0,  0,  0, 0,  0;    %UCLA
     1/3, 0,  0, 1/3, 0,  0,  0,   0,  0,  0, 0,  1;    %USC
     1/3, 0,  0,  0,  0,  0, .5,  1/3,1/3, 0, 0,  0;    %UT
     0,  1/3, 0, 1/3, 0,  0,  0,  1/3, 0, .5, 0,  0;    %WA
     0,   0,  0,  0, 1/3,.25,.5,   0,  0, .5, 0,  0;]   %WAS

 %create a regular stochastic matrix with google fudge factor:
SMb=.15/12*ones(12,12)+.85*SMa


 % the columns of a large power of SMb should be essentially identical,
 % and are the equilibrium solution.  The entries can be used to rank
 % the Pac 12 teams, based on current records.
SMb^20
```

Returning to 5.4-5.5:

$$[T]_\beta = P_{\beta \leftarrow E} \, [T]_E \, P_{E \leftarrow \beta}$$



<u>Exercise 1</u> (To review change of basis) Try to pick a better basis to understand the matrix transformation $T(\underline{\boldsymbol{x}}) = A\,\underline{\boldsymbol{x}}$, even though the matrix A is not diagonalizable. Compute $M = P^{-1} A\, P$ or compute M directly, to see if it really is a "better" matrix.

$$A = \begin{bmatrix} 4 & 4 \\ -1 & 0 \end{bmatrix}$$

5.5 Complex eigenvalues and eigenvectors.

We'll focus on 2×2 matrices, for simplicity. In this case it will turn out that a matrix with real entries and complex eigenvalues is always similar to a rotation-dilation matrix...

Definition A matrix of the form $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ is called a *rotation-dilation* matrix, because for

$r = \sqrt{a^2 + b^2}$ we can rewrite A as

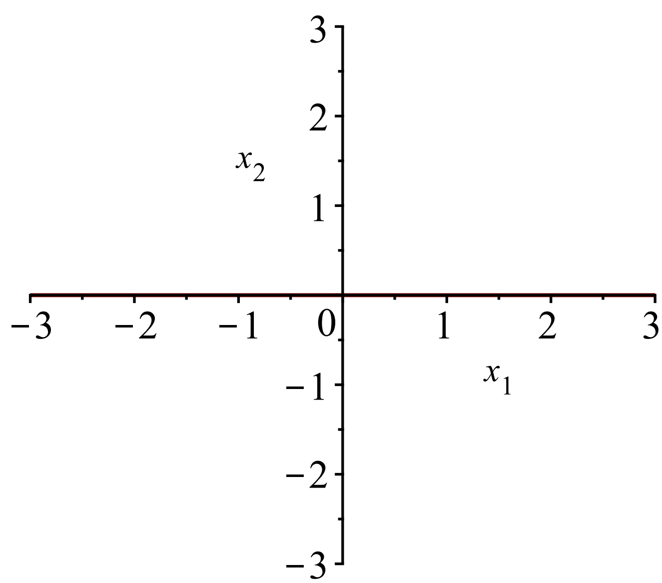
$$A = r \begin{bmatrix} \frac{a}{r} & -\frac{b}{r} \\ \frac{b}{r} & \frac{a}{r} \end{bmatrix} = r \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

So the transformation $T(\underline{\mathbf{x}}) = A\,\underline{\mathbf{x}}$ rotates vectors by an angle θ and then scales them by a factor of r. (So A^2 rotates by an angle 2θ and scales by r^2; A^3 rotates by an angle 3θ and scales by r^3, etc.

Exercise 2) Draw the transformation picture for

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

and interpret this transformation as a rotation-dilation. Overlay your diagram onto one picture: How about the transformation picture for $T \circ T = T^2$?



Exercise 3) What are the eigenvalues of a rotation-dilation matrix $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$? How do you think you would go about finding the eigenvectors?

Tues Nov 6

- 5.5 Complex eigendata

Announcements:

Warm-up Exercise:

It is possible for a matrix A with real entries to be diagonalizable if one allows complex scalars and vectors, even if it's not diagonalizable with real eigenvalues and eigenvectors. You saw an example of that on a food for thought problem, if you weren't afraid. We'll use a matrix today that we'll use later as well, in section 5.6, to study an interesting discrete dynamical system. This matrix is not a rotation-dilation matrix, but it is *similar* to one, and that fact will help us understand the discrete dynamical system.

Exercise 1) Let

$$B = \begin{bmatrix} .9 & -.4 \\ .1 & .9 \end{bmatrix}$$

Find the (complex) eigenvalues and eigenvectors for B.

General facts we saw illustrated in the example, about complex eigenvalues and eigenvectors: Let A be a matrix with real entries, and let

$$A\,\underline{\mathbf{v}} = \lambda\,\underline{\mathbf{v}}$$

with $\lambda = a + b\,i$, $\underline{\mathbf{v}} = \underline{\mathbf{u}} + i\,\underline{\mathbf{w}}$ complex, $(a, b \in \mathbb{R}, \underline{\mathbf{u}}, \underline{\mathbf{w}} \in \mathbb{R}^n)$. Then we write

$$\operatorname{Re}\lambda = a, \quad \operatorname{Im}\lambda = b$$

$$\operatorname{Re}\underline{\mathbf{v}} = \underline{\mathbf{u}}, \qquad \operatorname{Im}\underline{\mathbf{v}} = \underline{\mathbf{w}}.$$

So, the equation $A\,\underline{\mathbf{v}} = \lambda\,\underline{\mathbf{v}}$ expands as

$$A\,(\underline{\mathbf{u}} + i\,\underline{\mathbf{w}}) = (a + b\,i)\,(\underline{\mathbf{u}} + i\,\underline{\mathbf{w}}).$$

It will always be true then that the conjugate $\bar{\lambda} = a - b\,i$ is also an eigenvalue, and the conjugate vector $\bar{\underline{\mathbf{v}}} = \underline{\mathbf{u}} - i\,\underline{\mathbf{w}}$ will be a corresponding eigenvector, because it will satisfy

$$A\,(\underline{\mathbf{u}} - i\,\underline{\mathbf{w}}) = (a - b\,i)\,(\underline{\mathbf{u}} - i\,\underline{\mathbf{w}})$$

<u>Exercise 4</u> Verify that if the first eigenvector equation holds, then

$$A\,\underline{\mathbf{u}} = a\,\underline{\mathbf{u}} - b\,\underline{\mathbf{w}}$$
$$A\,\underline{\mathbf{w}} = b\,\underline{\mathbf{u}} + a\,\underline{\mathbf{w}}$$

Then check that these equalities automatically make the second conjugate eigenvector equation true as well.

Theorem Let A be a real 2×2 matrix with complex eigenvalues. Then A is similar to a rotation-dilation matrix.

proof: Let a complex eigenvalue and eigenvector be given by $\lambda = a + b\,i$, $\underline{\boldsymbol{v}} = \underline{\boldsymbol{u}} + i\,\underline{\boldsymbol{w}}$ complex, ($a, b \in \mathbb{R}$, $\underline{\boldsymbol{u}}, \underline{\boldsymbol{w}} \in \mathbb{R}^n$) Choose

$$P = [\ \mathrm{Re}\,\underline{\boldsymbol{v}} \quad \mathrm{Im}\,\underline{\boldsymbol{v}}] = [\underline{\boldsymbol{u}}\ \ \underline{\boldsymbol{w}}]$$

(One can check that $\{\underline{\boldsymbol{u}}, \underline{\boldsymbol{w}}\}$ is automatically independent.) Then, using the equations of Exercise 4, we mimic what we did for diagonalizable matrices...

$$A\,[\underline{\boldsymbol{u}}\ \ \underline{\boldsymbol{w}}] = [a\,\underline{\boldsymbol{u}} - b\,\underline{\boldsymbol{w}},\ b\,\underline{\boldsymbol{u}} + a\,\underline{\boldsymbol{w}}]$$

$$= [\ \underline{\boldsymbol{u}}\ \ \underline{\boldsymbol{w}}\ ]\begin{bmatrix} a & b \\ -b & a \end{bmatrix}.$$

$$A\,P = P\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

$$P^{-1}\,A\,P = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}.$$

(The matrix on the right is a rotation-dilation matrix ... nobody ever said what the sign of b was. :-))

It's a mess, but one can carry out the procedure of the theorem, for the matrix B in exercise 1,

$$B = \begin{bmatrix} .9 & -.4 \\ .1 & .9 \end{bmatrix}$$

using $\lambda = .9 - .2\,i$, $\underline{\boldsymbol{v}} = \begin{bmatrix} -2\,i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + i \begin{bmatrix} -2 \\ 0 \end{bmatrix}$, one gets

$$P = [\ \mathrm{Re}\,\underline{\boldsymbol{v}} \quad \mathrm{Im}\,\underline{\boldsymbol{v}}] = \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}, \qquad P^{-1} = \frac{1}{2}\begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}$$

$$P^{-1} B\, P = \frac{1}{2}\begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} .9 & -.4 \\ .1 & .9 \end{bmatrix}\begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}$$

$$P^{-1} B\, P = \begin{bmatrix} .9 & -.2 \\ .2 & .9 \end{bmatrix} = \sqrt{.85}\begin{bmatrix} \frac{.9}{\sqrt{.85}} & -\frac{.2}{\sqrt{.85}} \\ \frac{.2}{\sqrt{.85}} & \frac{.9}{\sqrt{.85}} \end{bmatrix}$$

$$P^{-1} B\, P = r\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

for $r = \sqrt{.85} \approx .92$, $\theta = \arctan\left(\frac{2}{9}\right) \approx .22$ radians.

Application on next page, to browse through...

72. Use the method outlined in Exercise 70 to check for which values of the constants a, b, and c the matrix

$$A = \begin{bmatrix} 1 & a & b \\ 0 & 0 & c \\ 0 & 0 & 1 \end{bmatrix} \text{ is diagonalizable.}$$

73. Prove the Cayley-Hamilton theorem, $f_A(A) = 0$, for diagonalizable matrices A. See Exercise 7.3.54.

74. In both parts of this problem, consider the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix},$$

with eigenvalues $\lambda_1 = 5$ and $\lambda_2 = -1$ (see Example 1).

a. Are the column vectors of the matrices $A - \lambda_1 I_2$ and $A - \lambda_2 I_2$ eigenvectors of A? Explain. Does this work for other 2×2 matrices? What about diagonalizable $n \times n$ matrices with two distinct eigenvalues, such as projections or reflections? (*Hint:* Exercise 70 is helpful.)

b. Are the column vectors of

$$A - \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

eigenvectors of A? Explain.

"Linear Algebra with Applications"
by Otto Bretscher

7.5 Complex Eigenvalues

Imagine that you are diabetic and have to pay close attention to how your body metabolizes glucose. After you eat a heavy meal, the glucose concentration will reach a peak, and then it will slowly return to the fasting level. Certain hormones help regulate the glucose metabolism, especially the hormone insulin. (Compare with Exercise 7.1.52.) Let $g(t)$ be the excess glucose concentration in your blood, usually measured in milligrams of glucose per 100 milliliters of blood. (*Excess* means that we measure how much the glucose concentration deviates from the fasting level.) A negative value of $g(t)$ indicates that the glucose concentration is below fasting level at time t. Let $h(t)$ be the excess insulin concentration in your blood. Researchers have developed mathematical models for the glucose regulatory system. The following is one such model, in slightly simplified (linearized) form.

$$\begin{aligned} g(t+1) &= ag(t) - bh(t) \\ h(t+1) &= cg(t) + dh(t) \end{aligned}$$

(These formulas apply between meals; obviously, the system is disturbed during and right after a meal.)

In these formulas, a, b, c, and d are positive constants; constants a and d will be less than 1. The term $-bh(t)$ expresses the fact that insulin helps your body absorb glucose, and the term $cg(t)$ represents the fact that the glucose in your blood stimulates the pancreas to secrete insulin.

For your system, the equations might be

$$\begin{aligned} g(t+1) &= 0.9g(t) - 0.4h(t) \\ h(t+1) &= 0.1g(t) + 0.9h(t), \end{aligned}$$

with initial values $g(0) = 100$ and $h(0) = 0$, after a heavy meal. Here, time t is measured in hours.

After one hour, the values will be $g(1) = 90$ and $h(1) = 10$. Some of the glucose has been absorbed, and the excess glucose has stimulated the pancreas to produce 10 extra units of insulin.

The rounded values of $g(t)$ and $h(t)$ in the following table give you some sense for the evolution of this dynamical system.

t	0	1	2	3	4	5	6	7	8	15	22	29
$g(t)$	100	90	77	62.1	46.3	30.6	15.7	2.3	−9.3	−29	1.6	9.5
$h(t)$	0	10	18	23.9	27.7	29.6	29.7	28.3	25.7	−2	−8.3	0.3

We can "connect the dots" to sketch a rough trajectory, visualizing the long-term behavior. See Figure 1.

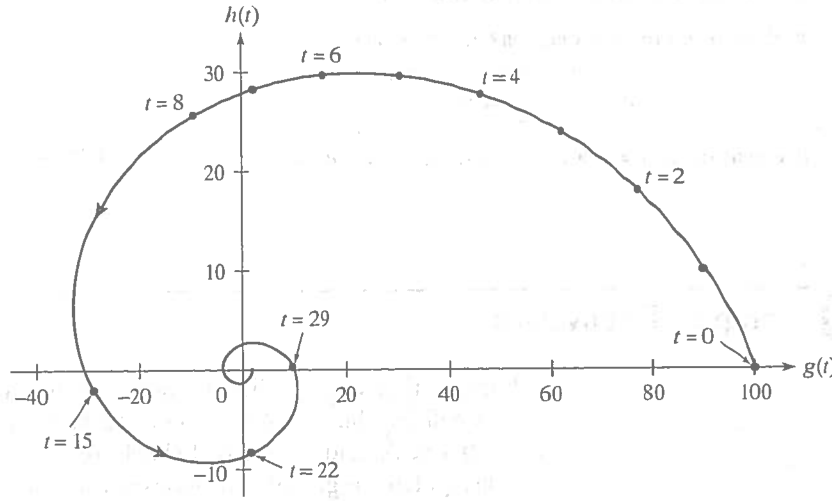


Figure 1

We see that after 7 hours the excess glucose is almost gone, but now there are about 30 units of excess insulin in the system. Since this excess insulin helps to reduce glucose further, the glucose concentration will now fall below fasting level, reaching about −30 after 15 hours. (You will feel awfully hungry by now.) Under normal circumstances, you would have taken another meal in the meantime, of course, but let's consider the case of (voluntary or involuntary) fasting.

We leave it to the reader to explain the concentrations after 22 and 29 hours, in terms of how glucose and insulin concentrations influence each other, according to our model. The *spiraling trajectory* indicates an *oscillatory behavior* of the system: Both glucose and insulin levels will swing back and forth around the fasting level, like a damped pendulum. Both concentrations will approach the fasting level (thus the name).

Another way to visualize this oscillatory behavior is to graph the functions $g(t)$ and $h(t)$ against time, using the values from our table. See Figure 2.

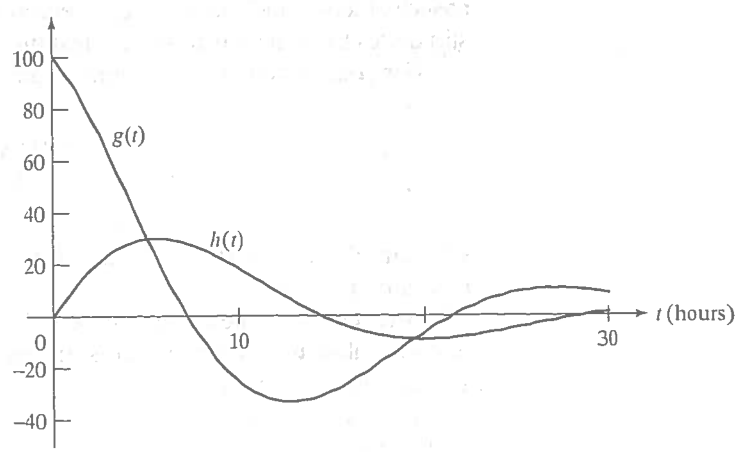


Figure 2

Example: From the algebra yesterday, and after a fair amount of work, For the dynamical system

$$\begin{bmatrix} g_{k+1} \\ h_{k+1} \end{bmatrix} = \begin{bmatrix} .9 & -.4 \\ .1 & .9 \end{bmatrix} \begin{bmatrix} g_k \\ h_k \end{bmatrix}$$

and with $\begin{bmatrix} g_0 \\ h_0 \end{bmatrix} = \begin{bmatrix} 100 \\ 0 \end{bmatrix}$, one can calculate and understand the spiral picture...

$$\begin{bmatrix} g_k \\ h_k \end{bmatrix} = \begin{bmatrix} .9 & -.4 \\ .1 & .9 \end{bmatrix}^k \begin{bmatrix} 100 \\ 0 \end{bmatrix} = = .92^k \begin{bmatrix} 100 \cos(k\theta) \\ 50 \sin(k\theta) \end{bmatrix}$$

$$\theta \approx .22 \text{ radians.}$$

yipes!

For $r = \sqrt{.85} \approx .92$, $\theta = \arctan\left(\frac{2}{9}\right) \approx .22$ radians.

$$B = r\,P \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} P^{-1}$$

$$B^2 = r^2\,P \begin{bmatrix} \cos(2\,\theta) & -\sin(2\,\theta) \\ \sin(2\,\theta) & \cos(2\,\theta) \end{bmatrix} P^{-1}$$

$$B^n = r^n\,P \begin{bmatrix} \cos(n\,\theta) & -\sin(n\,\theta) \\ \sin(n\,\theta) & \cos(n\,\theta) \end{bmatrix} P^{-1}$$

$$B^n \begin{bmatrix} 100 \\ 0 \end{bmatrix} = .92^n \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \cos(n\,\theta) & -\sin(n\,\theta) \\ \sin(n\,\theta) & \cos(n\,\theta) \end{bmatrix} \frac{1}{2} \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 100 \\ 0 \end{bmatrix}$$

$$.92^n \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \cos(n\,\theta) & -\sin(n\,\theta) \\ \sin(n\,\theta) & \cos(n\,\theta) \end{bmatrix} \begin{bmatrix} 0 \\ -50 \end{bmatrix}$$

$$= .92^n \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 50 \sin(n\theta) \\ -50 \cos(n\theta) \end{bmatrix}$$

$$= .92^n \begin{bmatrix} 100 \cos(n\theta) \\ 50 \sin(n\theta) \end{bmatrix}$$

Chapter 6 is about orthogonality and related topics. We'll spend maybe two weeks plus a day in this chapter. The ideas we develop start with the dot product, which we've been using algebraically to compute individual entries in matrix products, but which has important geometric meaning. By the end of the Chapter we will see applications to statistics, discuss generalizations of the dot product, "inner products", which can apply to function vector spaces and which lie at the heart of physics applications that use Fourier series, and more recent applications such as image and audio compression, see e.g.
https://en.wikipedia.org/wiki/Discrete_cosine_transform

Wed Nov 7

- 6.1-6.2 dot product, length, orthogonality, projection onto the span of a single vector.

Announcements:

Warm-up Exercise:

Recall, for any two vectors $\underline{v}, \underline{w} \in \mathbb{R}^n$, the dot product $\underline{v} \cdot \underline{w}$ is the scalar computed by the definition

$$\underline{v} \cdot \underline{w} := \sum_{i=1}^{n} v_i\, w_i \,.$$

We don't care if $\underline{v}, \underline{w}$ are row vectors or column vectors, or one of each, for the dot product.

We've been using the dot product algebraically to compute entries of matrix products $A\,B$, since

$$entry_{i\,j}\,[A\,B] = \left[row_i\,A\right]\left[col_j\,B\right] = \left[row_i\,A\right] \cdot \left[col_j\,B\right].$$

The algebra for dot products is a mostly a special case of what we already know for matrices, but worth writing down and double-checking, so we're ready to use it in the rest of Chapters 6 and 7.

Exercise 1 Check why

1a) dot product is commutative:

$$\underline{v} \cdot \underline{w} = \underline{w} \cdot \underline{v}\,.$$

1b) dot product distributes over addition:

$$(\underline{u} + \underline{v}) \cdot \underline{w} = \underline{u} \cdot \underline{w} + \underline{v} \cdot \underline{w}$$

$$\underline{u} \cdot (\underline{v} + \underline{w}) = \underline{u} \cdot \underline{v} + \underline{u} \cdot \underline{w}$$

1c) for $k \in \mathbb{R}$,

$$(k\,\underline{v}) \cdot \underline{w} = k\,(\underline{v} \cdot \underline{w}) = \underline{v} \cdot (k\,\underline{w}).$$

1d) dot product distributes over linear combinations:

$$\underline{v} \cdot \left(c_1\,\underline{w}_1 + c_2\,\underline{w}_2 + \ldots + c_k\,\underline{w}_k\right) = c_1\left(\underline{v} \cdot \underline{w}_1\right) + c_2\left(\underline{v} \cdot \underline{w}_2\right) + \ldots + c_k\left(\underline{v} \cdot \underline{w}_k\right).$$

$$\underline{\boldsymbol{v}} \cdot \underline{\boldsymbol{w}} := \sum_{i=1}^{n} v_i\, w_i$$

1e)

$$\underline{\boldsymbol{v}} \cdot \underline{\boldsymbol{v}} > 0 \quad \text{for each } \underline{\boldsymbol{v}} \neq \underline{\mathbf{0}} \quad (\text{and } \underline{\mathbf{0}} \cdot \underline{\mathbf{0}} = \underline{\mathbf{0}}.)$$

Chapter 6 is about topics related to the geometry of the dot product. It begins now, with definitions and consequences that generalize what you learned for $\mathbb{R}^2$, $\mathbb{R}^3$ in your multivariable Calculus class, to $\mathbb{R}^n$.

2) Geometry of the dot product, stage 1. We'll add examples with pictures as we go throught these definitions.

2a) For $\underline{\boldsymbol{v}} \in \mathbb{R}^n$ we define the *length* or *norm* or *magnitude* of $\underline{\boldsymbol{v}}$ by

$$\|\underline{\boldsymbol{v}}\| := \sqrt{\sum_{i=1}^{n} v_i^2} = (\underline{\boldsymbol{v}} \cdot \underline{\boldsymbol{v}})^{\frac{1}{2}} .$$

Notice that the length of a scalar multiple of a vector is what you'd expect:

$$\| t\,\underline{\boldsymbol{v}}\| = (t\,\underline{\boldsymbol{v}} \cdot t\,\underline{\boldsymbol{v}})^{\frac{1}{2}} = \left(t^2\, \underline{\boldsymbol{v}} \cdot \underline{\boldsymbol{v}}\right)^{\frac{1}{2}} = |\,t\,|\,\|\underline{\boldsymbol{v}}\| .$$

Also notice that $\|\underline{\boldsymbol{v}}\| > 0$ unless $\underline{\boldsymbol{v}} = \underline{\mathbf{0}}$.

2b) The distance between points (with position vectors) $\underline{\boldsymbol{P}}, \underline{\boldsymbol{Q}}$ is defined to be $\|\underline{\boldsymbol{Q}} - \underline{\boldsymbol{P}}\|$ (or $\|\underline{\boldsymbol{P}} - \underline{\boldsymbol{Q}}\|$).

2c) For $\underline{v}, \underline{w} \in \mathbb{R}^n$, we define $\underline{v}$ to be *orthogonal* (or *perpendicular*) to $\underline{w}$ if and only if

$$\underline{v} \cdot \underline{w} = 0.$$

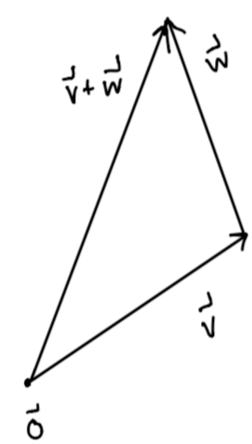
And in this case we write $\underline{v} \perp \underline{w}$.

Note: In $\mathbb{R}^2$ or $\mathbb{R}^3$ and in your multivariable calculus class, this definition was a special case of the identity

$$\underline{v} \cdot \underline{w} = \|\underline{v}\| \ \|\underline{w}\| \cos(\theta)$$

where θ is the angle between $\underline{v}, \underline{w}$. (Because $\cos(\theta) = 0$ when $\theta = \frac{\pi}{2}$.) That identity followed from the law of cosines, although you probably don't recall the details. In this class we'll actually use the identity above to *define* angles between vectors, in $\mathbb{R}^n$. (And in about two weeks, we can use it to define angles between functions, in inner product function spaces.)

2d) The $\mathbb{R}^n$ reason for defining orthogonality as in 2c is that the Pythagorean Theorem holds for the triangle with displacement vectors $\underline{v}, \underline{w}$ and hypotenuse $\underline{v} + \underline{w}$ if and only if $\underline{v} \cdot \underline{w} = 0$. Check!



2e) A vector $\underline{\boldsymbol{u}} \in \mathbb{R}^n$ is called a *unit vector* if and only if $\|\underline{\boldsymbol{u}}\| = 1$.

2f) If $\underline{\boldsymbol{v}} \in \mathbb{R}^n$ then the unit vector in the direction of $\underline{\boldsymbol{v}}$ is given by

$$\underline{\boldsymbol{u}} = \frac{1}{\|\underline{\boldsymbol{v}}\|}\underline{\boldsymbol{v}}.$$

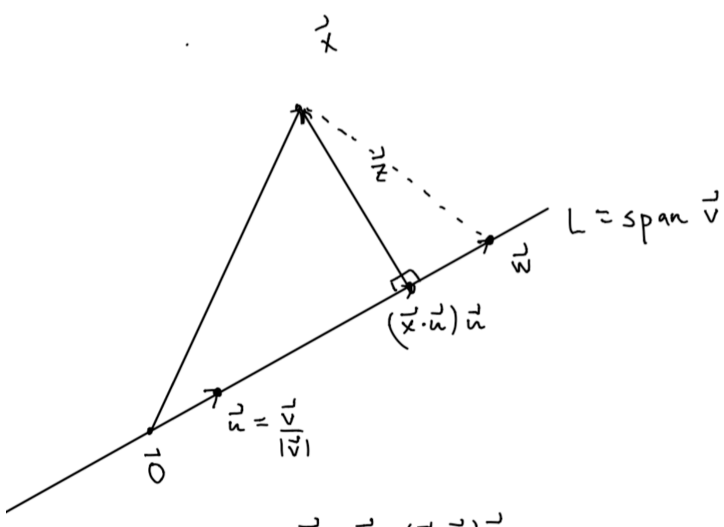
2g) Projection onto a line. Let $\underline{\boldsymbol{v}} \in \mathbb{R}^n$ be a non-zero vector, let $L = span\{\underline{\boldsymbol{v}}\}$ be a line through the origin. Then for any $\underline{\boldsymbol{x}} \in \mathbb{R}^n$ the projection of $\underline{\boldsymbol{x}}$ onto L is defined by the formula

$$proj_L\,\underline{\boldsymbol{x}} := (\underline{\boldsymbol{x}} \cdot \underline{\boldsymbol{u}})\,\underline{\boldsymbol{u}}$$

for $\underline{\boldsymbol{u}}$ the unit vector in the direction of $\underline{\boldsymbol{v}}$, $\underline{\boldsymbol{u}} = \frac{1}{\|\underline{\boldsymbol{v}}\|}\underline{\boldsymbol{v}}$. Equivalently

$$proj_L\,\underline{\boldsymbol{x}} := \frac{(\underline{\boldsymbol{x}} \cdot \underline{\boldsymbol{v}})}{\|\underline{\boldsymbol{v}}\|^2}\underline{\boldsymbol{v}}.$$

Then $proj_L\underline{\boldsymbol{x}}$ is the (position vector of) nearest point on L to (the point with position vector) $\underline{\boldsymbol{x}}$. To check why this is true use the diagram below. Show that $\underline{\boldsymbol{z}} := \underline{\boldsymbol{x}} - (\underline{\boldsymbol{x}} \cdot \underline{\boldsymbol{u}})\underline{\boldsymbol{u}}$ is perpendicular to $\underline{\boldsymbol{u}}$, so to any vector in $span\{\underline{\boldsymbol{u}}\}$. Then use the Pythagorean theorem to prove the claim.



$\vec{z}: \vec{x} - (\vec{x}\cdot\vec{u})\vec{u}$

$\vec{z}\cdot\vec{u} = (\vec{x} - (\vec{x}\cdot\vec{u})\vec{u})\cdot\vec{u} = (\vec{x}\cdot\vec{u}) - (\vec{x}\cdot\vec{u}) = 0$

So $\vec{z}\cdot t\vec{u} = 0\ \forall t$

for $\vec{w} \in L$, $\|\vec{x} - \vec{w}\|^2 = \|\vec{z}\|^2 + \|\vec{w} - (\vec{x}\cdot\vec{u})\vec{u}\|^2$

$\geq \|\vec{z}\|^2$

$= \|\vec{z}\|^2$ iff $\vec{w} = (\vec{x}\cdot\vec{u})\vec{u}$

2h) Refer to the same diagram as in 2g, which is an $\mathbb{R}^n$ picture. Using the Pythagorean triangle with edges $(\underline{\boldsymbol{x}} \cdot \underline{\boldsymbol{u}})\underline{\boldsymbol{u}}, \underline{\boldsymbol{z}}, \underline{\boldsymbol{x}}$ we have

$$\| (\underline{\boldsymbol{x}} \cdot \underline{\boldsymbol{u}})\underline{\boldsymbol{u}} \|^2 + \|\underline{\boldsymbol{z}}\|^2 = \|\underline{\boldsymbol{x}}\|^2.$$

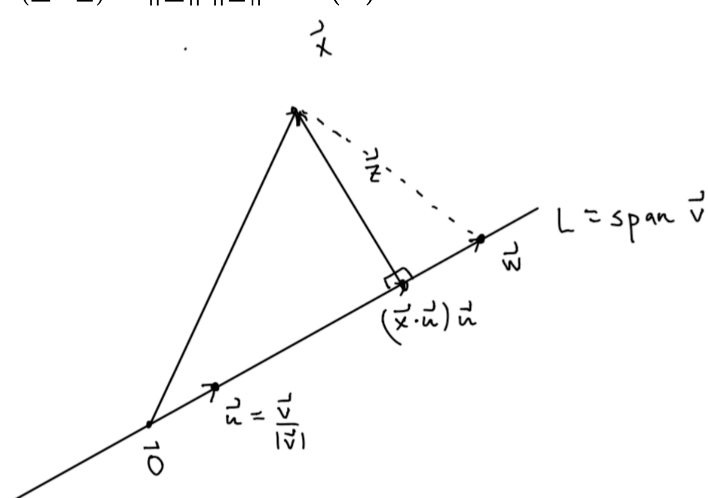
Define the angle θ between $\underline{\boldsymbol{v}}$ and $\underline{\boldsymbol{w}}$ the same way we would in $\mathbb{R}^2$, namely

$$\cos(\theta) = \frac{(\underline{\boldsymbol{x}} \cdot \underline{\boldsymbol{u}})}{\|\underline{\boldsymbol{x}}\|}.$$

Notice that because of the Pythagorean identity above, $-1 \leq \cos(\theta) \leq 1$, with $\cos(\theta) = 1$ if and only if $(\underline{\boldsymbol{x}} \cdot \underline{\boldsymbol{u}})\underline{\boldsymbol{u}} = \underline{\boldsymbol{x}}$ and $\cos(\theta) = -1$ if and only if $(\underline{\boldsymbol{x}} \cdot \underline{\boldsymbol{u}})\underline{\boldsymbol{u}} = -\underline{\boldsymbol{x}}$. So there is a unique θ with $0 \leq \theta \leq \pi$ for whic the cos θ equation can hold. Substituting $u = \frac{v}{\|v\|}$ gives the familiar formulas that you learned in multivariable Calculus for $\mathbb{R}^2$, $\mathbb{R}^3$, which now holds in $\mathbb{R}^n$.

$$\cos(\theta) = \frac{\left(\underline{\boldsymbol{x}} \bullet \frac{\underline{\boldsymbol{v}}}{\|\underline{\boldsymbol{v}}\|}\right)}{\|\underline{\boldsymbol{x}}\|} = \frac{(\underline{\boldsymbol{x}} \bullet \underline{\boldsymbol{v}})}{\|\underline{\boldsymbol{x}}\| \, \|\underline{\boldsymbol{v}}\|}, \text{ i.e.}$$

$$(\underline{\boldsymbol{x}} \bullet \underline{\boldsymbol{v}}) = \|\underline{\boldsymbol{x}}\| \, \|\underline{\boldsymbol{v}}\| \cos(\theta)$$



$\vec{z}$: $\vec{x} - (\vec{x}\cdot\vec{u})\vec{u}$

$\vec{z}\cdot\vec{u} = (\vec{x} - (\vec{x}\cdot\vec{u})\vec{u})\cdot\vec{u} = (\vec{x}\cdot\vec{u}) - (\vec{x}\cdot\vec{u}) = 0$

So $\vec{z} \cdot t\vec{u} = 0 \ \forall t$

for $\vec{w} \in L$, $\|\vec{x}-\vec{w}\|^2 = \|\vec{z}\|^2 + \|\vec{w} - (\vec{x}\cdot\vec{u})\vec{u}\|^2$

$\geq \|\vec{z}\|^2$

$= \|\vec{z}\|^2$ iff $\vec{w} = (\vec{x}\cdot\vec{u})\vec{u}$

3) <u>Summary exercise</u> In $\mathbb{R}^2$, let $L = span\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$. Find $proj_L \begin{bmatrix} 3 \\ 4 \end{bmatrix}$. Illustrate. Verify the Pythagorean Theorem for $proj_L \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, "$\underset{\sim}{z}$" and hypotenuse $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$.