Exercise 3a) Why must more than three vectors in $\mathbb{R}^3$ be linearly dependent?

consider $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + c_4\vec{v}_4 + \dots + c_n\vec{v}_n = \vec{0}$ $\quad n \geq 4$ vectors

augmented matrix

$$3\left\{\underbrace{\left[\begin{array}{cccc|c} \vec{v}_1 & \vec{v}_2 & & \vec{v}_n & 0 \\ & & & & 0 \\ & & & & 0 \end{array}\right]}_{n}\right. \longrightarrow \underbrace{\left[\begin{array}{c|c} & 0 \\ & 0 \\ & 0 \end{array}\right]}$$

at most 3 pivots
at least $n-3$ free variables
so, lots of dependencies.

3b) How about more than $m$ vectors in $\mathbb{R}^m$ ?

$$m\left\{\underbrace{\left[\begin{array}{cccc} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ & & & \\ & & & \end{array}\right]}_{n>m}\right.$$

at most $m$ pivots
at least $n-m$ free variables
so, lots of dependencies

3c) If you are given a set of exactly $n$ vectors in $\mathbb{R}^n$ how can you check whether or not they are linearly independent?

$$n\left\{\underbrace{\left[\begin{array}{c|c|c|c} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ & & & \\ & & & \end{array}\right]}_{n}\right.$$

$c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n = \vec{0}$

rref

$$\begin{bmatrix} 1 & 0 & 0 & - \\ 0 & 1 & 0 & - \\ \vdots & & & \\ 0 & \cdot & - - 0 & 1 \end{bmatrix} \text{ "} I \text{"}$$ vectors independent

$< n$ pivots
so non-pivot cols, so free variables
so dependent

3d) If you have a set of fewer than $n$ vectors in $\mathbb{R}^n$ can they span $\mathbb{R}^n$?

Friday warm-up

NO

$p<n$: $\quad x_1\vec{v}_1 + x_2\vec{v}_2 + \dots + x_p\vec{v}_p = \vec{b} \in \mathbb{R}^n$

$$n\left\{\underbrace{\left[\begin{array}{cccc|c} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_p & \vec{b} \end{array}\right]}_{p<n}\right. \longrightarrow \left[\begin{array}{c|c} & c_1 \\ & c_2 \\ & \vdots \\ 0\ 0\ 0 \cdots 0 & c_n \end{array}\right]$$

inconsistent unless $c_n = 0$.

at most $p$ pivots in coeff matrix $p<n$, so bottom row of coeff. matrix reduces to 0

3e) If $\{\underline{v}_1, \underline{v}_2, \dots \underline{v}_n\}$ is a set of exactly $n$ vectors in $\mathbb{R}^n$ what condition on the reduced row echelon form of $A = [\underline{v}_1, \underline{v}_2, \dots \underline{v}_n]$ guarantees and is required so that the vectors span $\mathbb{R}^n$ ? Compare with 3c.

Friday warm-up

if exactly $n$ vectors in $\mathbb{R}^n$

need $\operatorname{rref} A = I$ in order for the vectors to span $\mathbb{R}^n$
(else there's a bottom row of 0's in rref(A)).

Fri Sept 7

- 1.7 reduced row echelon form as encoding linear independence/dependence of matrix columns; introduction to linear transformations, section 1.8.

Announcements: returned HW 1 & Quiz 3

only some of the problems are graded
- see rubric on public HW page

solutions to non-text problems are on CANVAS.

'til 12:57

Warm-up Exercise: 3d, 3e; the last two exercises in yesterday's notes

Exercise 1) Consider the homogeneous matrix equation $A\,\underline{\boldsymbol{x}} = \underline{\mathbf{0}}$, with the matrix $A$ (and its reduced row echelon form) shown below: $A_{3\times 5}$ variables $x_1, x_2, \ldots x_5$

$$A := \begin{array}{c} \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 1 & 1 & 0 \\ 2 & 4 & 1 & 4 & 1 & 0 \\ -2 & -4 & 0 & -2 & -2 & 0 \end{array}\right] \\ \vec{a}_1 \;\; \vec{a}_2 \;\; \vec{a}_3 \;\; \vec{a}_4 \;\; \vec{a}_5 \end{array} \rightarrow \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

in class I accidentally used "$\vec{a}$" for the columns, which disagreed with how I named them here

Find and express the solutions to this system in linear combination form. Note that you are finding all of the dependencies for the collection of vectors that are the columns of $A$, namely the set

$$\left\{ \vec{a}_1 = \underline{\boldsymbol{v}}_1 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}, \vec{a}_2 = \underline{\boldsymbol{v}}_2 = \begin{bmatrix} 2 \\ 4 \\ -4 \end{bmatrix}, \vec{a}_3 = \underline{\boldsymbol{v}}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \vec{a}_4 = \underline{\boldsymbol{v}}_4 = \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}, \vec{a}_5 = \underline{\boldsymbol{v}}_5 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \right\}.$$

$x_1 = -2x_2 - x_4 - x_5$
$x_2 = x_2$ free
$x_3 = -2x_4 + x_5$
$x_4 = x_4$ free
$x_5 = x_5$ free

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

these homog. sol'tns also encode column dependencies

e.g. $x_2 = 1,\ x_4 = x_5 = 0 \Rightarrow -2\vec{a}_1 + 1\,\vec{a}_2 = \vec{0}$ $\quad \vec{v}_2 = 2\vec{v}_1$ next page

$$-2\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} + 1 \cdot \begin{bmatrix} 2 \\ 4 \\ -4 \end{bmatrix} = \vec{0} \checkmark$$

e.g. $x_2 = x_5 = 0,\ x_4 = 1 \Longrightarrow -\vec{a}_1 - 2\vec{a}_3 + \vec{a}_4 = \vec{0}$

e.g. $x_2 = x_4 = 0,\ x_5 = 1 \Rightarrow -\vec{a}_1 + \vec{a}_3 + \vec{a}_5 = \vec{0}$

$\vec{v}_4 = \vec{v}_1 + 2\vec{v}_3$ next page

$\vec{v}_5 = \vec{v}_1 - \vec{v}_3$ next page

Exercise 2) Focus on the idea that solutions to homogeneous matrix equations correspond exactly to linear dependencies between the columns of the matrix. Now, since the solution set to a homogeneous linear system does not change as you do elementary row operations to the augmented matrix, column dependencies also do not change. Therefore the vectors that span the space of homogeneous solutions in Exercise 1 are encoding the key column dependencies in $\mathbb{R}^3$, for both the original matrix, and for the reduced row echelon form.

Check this, by reading off "easy" column dependencies in the reduced matrix; seeing that they are also dependencies in the original matrix; and that they correspond to the basis of the homogeneous solution space. Magic! We will use this magic in important interesting ways, later in the course.

$$A := \begin{array}{c} \begin{bmatrix} 1 & 2 & 0 & 1 & 1 \\ 2 & 4 & 1 & 4 & 1 \\ -2 & -4 & 0 & -2 & -2 \end{bmatrix} \\ \begin{array}{ccccc} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 & \vec{v}_5 \end{array} \end{array} \rightarrow \begin{array}{c} \begin{bmatrix} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{ccccc} \vec{w}_1 & \vec{w}_2 & \vec{w}_3 & \vec{w}_4 & \vec{w}_5 \end{array} \end{array}$$

$\vec{v}_2 = 2\vec{v}_1 \quad \Longleftarrow \quad \vec{w}_2 = 2\vec{w}_1 \qquad \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\vec{v}_1, \vec{v}_3$ are ind. $\quad \Longleftarrow \quad \vec{w}_1, \vec{w}_3$ are independent

$\vec{v}_4 = \vec{v}_1 + 2\vec{v}_3 \quad \Longleftarrow \quad \vec{w}_4 = \vec{w}_1 + 2\vec{w}_3 \qquad \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\vec{v}_5 = \vec{v}_1 - \vec{v}_3 \quad \Longleftarrow \quad \vec{w}_5 = \vec{w}_1 - \vec{w}_3$

compare to previous page

Exercise 3) (This exercise explains why each matrix has only one reduced row echelon form, no matter what sequence of elementary row operations one uses to find it. We didn't have the tools to explain why this fact was true earlier in the chapter) Let $B_{4 \times 5}$ be a matrix whose columns satisfy the following dependencies:

$$col_1(B) \neq \mathbf{0} \quad \text{(i.e. is independent)}$$
$$col_2(B) = 3\, col_1(B)$$
$$col_3(B) \text{ is independent of column 1}$$
$$col_4(B) \text{ is independent of columns 1,3.}$$
$$col_5(B) = -3\, col_1(B) + 2\, col_3(B) - col_4(B).$$

What is the reduced row echelon form of $B$? $B$ 4 rows, 5 columns

$$B = [\vec{b}_1\ \vec{b}_2\ \vec{b}_3\ \vec{b}_4\ \vec{b}_5] \Longrightarrow rref(B) = \begin{bmatrix} 1 & 3 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$-3\begin{bmatrix}1\\0\\0\\0\end{bmatrix} + 2\begin{bmatrix}0\\1\\0\\0\end{bmatrix} - 1\begin{bmatrix}0\\0\\1\\0\end{bmatrix}$$

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What if:
$$\begin{matrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}$$

would violate rref conditions!

depending on $5^{th}$ column would either violate
* zero rows at bottom

or * pivots move to right as you go down the rows

1.8 Introduction to linear transformations.

F

Definition: A function $T$ which has domain equal to $\mathbb{R}^n$ and whose range lies in $\mathbb{R}^m$ is called a *linear transformation* if it transforms sums to sums, and scalar multiples to scalar multiples. Precisely, $T : \mathbb{R}^n \to \mathbb{R}^m$ is *linear* if and only if

$$T(\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}}) = T(\underline{\boldsymbol{u}}) + T(\underline{\boldsymbol{v}}) \qquad \forall\ \underline{\boldsymbol{u}}, \underline{\boldsymbol{v}} \in \mathbb{R}^n$$
$$T(c\,\underline{\boldsymbol{u}}) = c\,T(\underline{\boldsymbol{u}}) \qquad \forall\ c \in \mathbb{R}, \underline{\boldsymbol{u}} \in \mathbb{R}^n\ .$$

Notation In this case we call $\mathbb{R}^m$ the *codomain*. We call $T(\underline{\boldsymbol{u}})$ the *image of* $\underline{\boldsymbol{u}}$. The *range* of $T$ is the collection of all images $T(\underline{\boldsymbol{u}})$, for $\underline{\boldsymbol{u}} \in \mathbb{R}^n$.

Important connection to matrices: Each matrix $A_{m \times n}$ gives rise to a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, namely

$$T(\underline{\boldsymbol{x}}) := A\,\underline{\boldsymbol{x}} \qquad \forall\ \underline{\boldsymbol{x}} \in \mathbb{R}^n.$$

e.g.

$$\begin{bmatrix} -1 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \qquad \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

This is because

Theorem 5 (p. 39 Matrix multiplication is linear) If $A$ is an $m \times n$ matrix, $\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}} \in \mathbb{R}^n$, $c$ a scalar, then

a) $A(\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}}) = A\,\underline{\boldsymbol{u}} + A\,\underline{\boldsymbol{v}}$

e.g. $n = 3$ columns

$$A(\vec{u}+\vec{v}) = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{bmatrix} \begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ u_3+v_3 \end{bmatrix} = (u_1+v_1)\vec{a}_1 + (u_2+v_2)\vec{a}_2 + (u_3+v_3)\vec{a}_3$$
$$= u_1\vec{a}_1 + u_2\vec{a}_2 + u_3\vec{a}_3 + v_1\vec{a}_1 + v_2\vec{a}_2 + v_3\vec{a}_3$$
$$= A\vec{u} + A\vec{v}$$

b) $A(c\,\underline{\boldsymbol{u}}) = c\,A\,\underline{\boldsymbol{u}}$

Remark: One reason that the word "linear" is appropriate for these sorts of functions is that linear transformations transform lines to lines (or points); and families of parallel lines are transformed into families of parallel lines (or points).

to be continued!