Wed Sept 5

- 1.7 Linear dependence/independence continued.

Announcements:
- quiz today
- usually you'll get back all assignments within a week (or sooner). HW delayed this week
- HW3 is mostly posted. All by tonight

Warm-up Exercise: Exercise 1 in today's notes

$c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n = \vec{0}$ has non-zero solutions $\vec{c}$

Note that the set of vectors $\{\underline{v}_1, \underline{v}_2, \ \dots \ \underline{v}_n\}$ is linearly dependent if and only if there are non-zero solutions $\underline{c}$ to the homogeneous matrix equation

$$A\,\underline{c} = \underline{0}$$

for the matrix $A = [\underline{v}_1 \ \underline{v}_2 \ \dots \ \underline{v}_n]$ having the given vectors as columns. Thus all linear independence/dependence questions can be answered using reduced row echelon form and facts about homogeneous matrix solutions.

Warmup problem

Exercise 1) Show that the vectors

$$\underline{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \ \underline{v}_2 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \ \underline{v}_3 = \begin{bmatrix} -1 \\ 6 \\ 4 \end{bmatrix}$$

are linearly dependent (even though no two of them are scalar multiples of each other). What does this mean geometrically about the span of these three vectors?

to solve

$$c_1\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + c_2\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + c_3\begin{bmatrix} -1 \\ 6 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

are there sol'ns besides $c_1 = c_2 = c_3 = 0$?

$\boxed{\text{Answer: dependent}}$

Hint: You might find this computation useful:

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 2 & 6 & 0 \\ 2 & 0 & 4 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

there are non-trivial solutions

- c_3 is a free variable (∞'ly many sol'ns)

more specific:

$c_1 = -2c_3$
$c_2 = -3c_3$
$c_3 =$ free

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = c_3\begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}.$$

e.g. $-2\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - 3\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 6 \\ 4 \end{bmatrix} = \vec{0}$

OR!

$c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{v}_3$?

$$\left[\begin{array}{cc|c} 1 & -1 & -1 \\ 0 & 2 & 6 \\ 2 & 0 & 4 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right]$$

so $2\vec{v}_1 + 3\vec{v}_2 = \vec{v}_3$

Exercise 2) Are the vectors

$$\underline{\boldsymbol{v}}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \ \underline{\boldsymbol{v}}_2 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \ \underline{\boldsymbol{w}}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

linearly independent or dependent? Hint:

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ 2 & 0 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right].$$

vector eqtn

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0} \Rightarrow \begin{array}{l} c_1 = 0 \\ c_2 = 0 \\ c_3 = 0 \end{array}$$

Independent

Exercise 3a) Why must more than three vectors in $\mathbb{R}^3$ be linearly dependent?

consider $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + c_4\vec{v}_4 + \dots + c_n\vec{v}_n = \vec{0}$ $\quad n \geq 4$ vectors

augmented matrix $3\left\{\underbrace{\left[\begin{array}{cccc|c} \vec{v}_1 & \vec{v}_2 & & \vec{v}_n & 0 \\ & & & & 0 \\ & & & & 0 \end{array}\right]}_{n}\right. \longrightarrow \underbrace{\left[\begin{array}{c|c} & 0 \\ & 0 \\ & 0 \end{array}\right]}$

at most 3 pivots
at least $n-3$ free variables
so, lots of dependencies.

3b) How about more than m vectors in $\mathbb{R}^m$?

$m\left\{\underbrace{\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ & & & \end{bmatrix}}_{n > m}\right.$

at most m pivots
at least $n-m$ free variables
so, lots of dependencies

3c) If you are given a set of exactly n vectors in $\mathbb{R}^n$ how can you check whether or not they are linearly independent?

$n\left\{\underbrace{\left[\begin{array}{c|c|c|c} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ & & & \end{array}\right]}_{n}\right.$ rref $\begin{bmatrix} 1 & 0 & 0 & - \\ 0 & 1 & 0 & - \\ \vdots & & & \\ 0 & \cdots & 0 & 1 \end{bmatrix}$ "I" vectors independent

$< n$ pivots
so non-pivot cols, so free variables
so dependent

$c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n = \vec{0}$

3d) If you have a set of fewer than n vectors in $\mathbb{R}^n$ can they span $\mathbb{R}^n$?

3e) If $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \dots \underline{\boldsymbol{v}}_n\}$ is a set of exactly n vectors in $\mathbb{R}^n$ what condition on the reduced row echelon form of $A = [\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \dots \underline{\boldsymbol{v}}_n]$ guarantees and is required so that the vectors span $\mathbb{R}^n$? Compare with 3c.