Math 2270-002 Week 3 notes
We will not necessarily finish the material from a given day's notes on that day. We may also add or subtract some material as the week progresses, but these notes represent an outline of what we plan to cover. These notes cover material in 1.6-1.8.

Tues ~~Sept 4~~
- 1.6-1.7 applications; introduction to linear dependence and independence.

Announcements: part of next week's HW:
- §1.7 1, 5, 7, 13, 17, 21 (more later)
- HW due tomorrow (office hours after class today LCB 225 (MTW))
- Quiz tomorrow

Warm-up Exercise: Preview applications exercises 1 & 2, taken from text §1.6, in today's notes network problems $\rightarrow$ linear algebra

Exercise 1) Consider the following traffic flow problem (from our text): What are the possible flow patterns, based on the given information and that the streets are one-way, so none of the flow numbers can be negative?

EXAMPLE 2 The network in Figure 2 shows the traffic flow (in vehicles per hour) over several one-way streets in downtown Baltimore during a typical early afternoon. Determine the general flow pattern for the network.

flow in = flow out

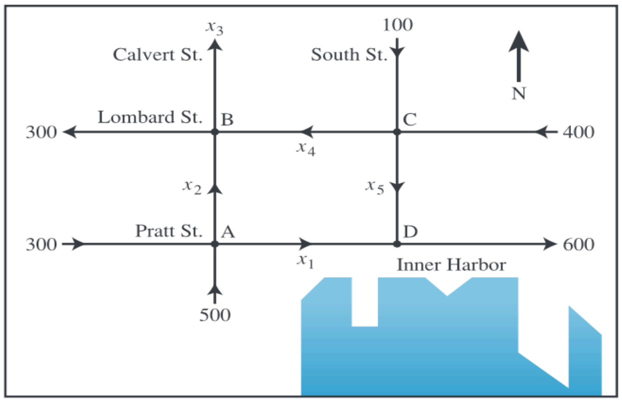

FIGURE 2 Baltimore streets.

A $\quad 800 = x_1 + x_2$

B $\quad x_2 + x_4 = 300 + x_3$

C $\quad 500 = x_4 + x_5$

D $\quad x_1 + x_5 = 600$

$$\begin{aligned} x_1 + x_2 \qquad\qquad &= 800 \\ x_2 - x_3 + x_4 \quad &= 300 \\ x_4 + x_5 &= 500 \\ x_1 \qquad\qquad + x_5 &= 600 \end{aligned}$$

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 800 \\ 0 & 1 & -1 & 1 & 0 & 300 \\ 0 & 0 & 0 & 1 & 1 & 500 \\ 1 & 0 & 0 & 0 & 1 & 600 \end{array}\right]$$

Hint: If you set up the flow equations for intersections A, B, C, D in that order, the following reduced row echelon form computation may be helpful:

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 800 \\ 0 & 1 & -1 & 1 & 0 & 300 \\ 0 & 0 & 0 & 1 & 1 & 500 \\ 1 & 0 & 0 & 0 & 1 & 600 \end{array}\right] \rightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 1 & 600 \\ 0 & 1 & 0 & 0 & -1 & 200 \\ 0 & 0 & 1 & 0 & 0 & 400 \\ 0 & 0 & 0 & 1 & 1 & 500 \end{array}\right]$$

$x_1 = 600 - x_5$
$x_2 = 200 + x_5$
$x_3 = 400$
$x_4 = 500 - x_5$
$x_5 =$ free

Note: $0 \le x_5 \le 500$ since all flow rates ≥ 0

Exercise 2)

A Homogeneous System in Economics

WEB

The system of 500 equations in 500 variables, mentioned in this chapter's introduction, is now known as a Leontief "input–output" (or "production") model.[1] Section 2.6 will examine this model in more detail, when more theory and better notation are available. For now, we look at a simpler "exchange model," also due to Leontief.

Suppose a nation's economy is divided into many sectors, such as various manufacturing, communication, entertainment, and service industries. Suppose that for each sector we know its total output for one year and we know exactly how this output is divided or "exchanged" among the other sectors of the economy. Let the total dollar value of a sector's output be called the **price** of that output. Leontief proved the following result.

> There exist *equilibrium prices* that can be assigned to the total outputs of the various sectors in such a way that the income of each sector exactly balances its expenses.

The following example shows how to find the equilibrium prices.

EXAMPLE 1 Suppose an economy consists of the Coal, Electric (power), and Steel sectors, and the output of each sector is distributed among the various sectors as shown in Table 1, where the entries in a column represent the fractional parts of a sector's total output.

The second column of Table 1, for instance, says that the total output of the Electric sector is divided as follows: 40% to Coal, 50% to Steel, and the remaining 10% to Electric. (Electric treats this 10% as an expense it incurs in order to operate its

[1] See Wassily W. Leontief, "Input–Output Economics," *Scientific American*, October 1951, pp. 15–21.

business.) Since all output must be taken into account, the decimal fractions in each column must sum to 1.

Denote the prices (i.e., dollar values) of the total annual outputs of the Coal, Electric, and Steel sectors by p_C, p_E, and p_S, respectively. If possible, find equilibrium prices that make each sector's income match its expenditures.

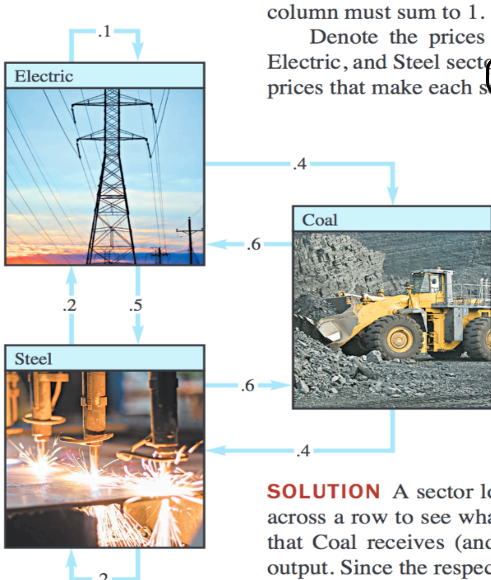

TABLE 1 A Simple Economy

Distribution of Output from:			
Coal	Electric	Steel	Purchased by:
.0	.4	.6	Coal
.6	.1	.2	Electric
.4	.5	.2	Steel

SOLUTION A sector looks down a column to see where its output goes, and it looks across a row to see what it needs as inputs. For instance, the first row of Table 1 says that Coal receives (and pays for) 40% of the Electric output and 60% of the Steel output. Since the respective values of the total outputs are p_E and p_S, Coal must spend $.4p_E$ dollars for its share of Electric's output and $.6p_S$ for its share of Steel's output. Thus Coal's total expenses are $.4p_E + .6p_S$. To make Coal's income, p_C, equal to its expenses, we want

$$p_C = .4p_E + .6p_S \tag{1}$$

$p_c = .4p_E + .6p_s$ $\quad -p_c + .4p_E + .6p_s = 0$ $\quad$ $\left.\begin{array}{ccc|c} -1 & .4 & .6 & 0 \\ .6 & -.9 & .2 & 0 \\ .4 & .5 & -.8 & 0 \end{array}\right.$

$p_E = .6p_c + .1p_E + .2p_s$

$p_s = .4p_c + .5p_E + .2p_s$

Hint: After we set up the problem, the following computation will help with the answer:

$$\left[\begin{array}{ccc|c} -1. & .4 & .6 & 0 \\ .6 & -.9 & .2 & 0 \\ .4 & .5 & -.8 & 0 \end{array}\right] \text{ approximately reduces to } \begin{array}{c} \begin{array}{ccc} p_c & p_E & p_s \end{array} \\ \left[\begin{array}{ccc|c} 1. & -0. & -0.94 & 0 \\ 0. & 1. & -0.85 & 0 \\ 0. & 0. & 0. & 0 \end{array}\right] \end{array}$$

$$p_c = .94\, p_s$$
$$p_E = .85\, p_s$$
$$p_s = \text{free}$$

$$\begin{bmatrix} p_c \\ p_E \\ p_s \end{bmatrix} = p_s \begin{bmatrix} .94 \\ .85 \\ 1 \end{bmatrix}$$

§1.6 just gives a tiny taste of applications linear independence & dependence for sets of vectors.

1.7 When we are discussing the span of a collection of vectors $\underline{v}_1, \underline{v}_2, \ldots \underline{v}_n$ we would like to know that we are being efficient in describing this span, and not wasting any free parameters because of redundancies in the vectors. For example, the most efficient way to describe a plane in $\mathbb{R}^3$ is as the span of exactly two vectors, rather than as the span of three or more. This has to do with the concept of "linear independence":

logical opposite of linearly dependent

Definition:

a) An indexed set of vectors $\{\underline{v}_1, \underline{v}_2, \ldots \underline{v}_n\}$ is said to be linearly independent if no one of the vectors is a linear combination of (some) of the other vectors. The concise way to say this is that the only way $\underline{\mathbf{0}}$ can be expressed as a linear combination of these vectors,

$$c_1\underline{v}_1 + c_2\underline{v}_2 + \ldots + c_n\underline{v}_n = \underline{\mathbf{0}},$$

1 opposite of 1 for dependent

is for all of the weights $c_1 = c_2 = \ldots = c_n = 0$.

2

logical negation of 2 for dependent

start here.

2 one $\vec{v}_j$ is a linear combo of (some) of the rest

b) An indexed set of vectors $\{\underline{v}_1, \underline{v}_2, \ldots \underline{v}_n\}$ is said to be linearly dependent if at least one of these vectors *is* a linear combination of (some) of the other vectors. The concise way to say this is that there *is* some way to write $\underline{\mathbf{0}}$ as a linear combination of these vectors

$$(1)\quad c_1\underline{v}_1 + c_2\underline{v}_2 + \ldots + c_n\underline{v}_n = \underline{\mathbf{0}}$$

where *not all* of the $c_j = 0$. (We call such an equation a linear dependency. Note that if we have any such linear dependency, then any $\underline{v}_j$ with $c_j \neq 0$ is a linear combination of the remaining $\underline{v}_k$ with $k \neq j$. We say that such a $\underline{v}_j$ is linearly dependent on the remaining $\underline{v}_k$.)

If 1 is true, $c_j \neq 0$. then solve for $\vec{v}_j$ in 2

$$c_j\vec{v}_j = \underbrace{-c_1\vec{v}_1 - c_2\vec{v}_2 - \ldots - c_n\vec{v}_n}_{\text{no } \vec{v}_j \text{ terms.}}$$

so $\vec{v}_j = -\frac{c_1}{c_j}\vec{v}_1 - \frac{c_2}{c_j}\vec{v}_2 - \ldots - \frac{c_n}{c_j}\vec{v}_n.$

If 2 is true, i.e. $\vec{v}_j = \underbrace{d_1\vec{v}_1 + d_2\vec{v}_2 + \ldots + d_n\vec{v}_n}_{\text{no } \vec{v}_j\text{'s.}}$

then $\vec{0} = d_1\vec{v}_1 + d_2\vec{v}_2 + \ldots + d_n\vec{v}_n - \vec{v}_j$ i.e. 1 is true.

Note: The only set of a single vector $\{\underline{\boldsymbol{v}}_1\}$ that is dependent is if $\underline{\boldsymbol{v}}_1 = \underline{\mathbf{0}}$. The only sets of two non-zero vectors, $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2\}$ that are linearly dependent are when one of the vectors is a scalar multiple of the other one. For more than two vectors the situation is more complicated.

if dep. $\Rightarrow c_1\vec{v}_1 = \vec{0}$ $\quad c_1 \neq 0$

$\Rightarrow \vec{v}_1 = \vec{0}$.

& for the zero vector $\vec{0}$

$1 \cdot \vec{0} = \vec{0}$

is a dependency.

If $\{\vec{v}_1, \vec{v}_2\}$ linearly dependent

either $\vec{v}_2 = c_1\vec{v}_1$

(or $\vec{v}_1 = c_2\vec{v}_2$)

So one of them is a multiple of the other.

Exercise 3a) Is this set linearly dependent or independent? $\left\{\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\}$? How does your answer generalize to any set of vectors which includes the zero vector?

dependent

$1\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

e.g. $\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\}$

$0\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 1 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

dependent.

3b) Is this set of vectors linearly dependent or independent? $\left\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ -3 \end{bmatrix}\right\}$?

$\vec{v}_1 \qquad \vec{v}_2$

$$\left.\begin{aligned} -3\vec{v}_1 &= \vec{v}_2 \\ 3\vec{v}_1 + \vec{v}_2 &= \vec{0} \\ \text{or} \quad -3\vec{v}_1 - \vec{v}_2 &= \vec{0} \\ 6\vec{v}_1 + 2\vec{v}_2 &= \vec{0} \end{aligned}\right\} \text{ any one suffices}$$

here after def of dependence

Example

The set of vectors $\left\{ \underline{\boldsymbol{v}}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \underline{\boldsymbol{v}}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \underline{\boldsymbol{v}}_3 = \begin{bmatrix} -2 \\ 8 \end{bmatrix} \right\}$ in $\mathbb{R}^2$ is linearly dependent because, as we showed when we were introducing vector equations (and as we can quickly recheck),

$$-3.5\begin{bmatrix} 1 \\ -1 \end{bmatrix} + 1.5\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 8 \end{bmatrix}.$$

$\vec{v}_3 = -3.5\,\vec{v}_1 + 1.5\,\vec{v}_2$ 2

We can also write this linear dependency as

$$-3.5\underline{\boldsymbol{v}}_1 + 1.5\underline{\boldsymbol{v}}_2 - \underline{\boldsymbol{v}}_3 = \underline{\mathbf{0}}$$

1 dependent

(or any non-zero multiple of that equation.)

two ways of showing dependent

but $\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$ is independent

Remark: If we're studying the span of a set of vectors, we'd prefer to be dealing with independent ones in order to avoid redundancies in how we represent a given vector as a linear combination. If you look at the example above, we can delete the vector v_3 (or any one of the other two vectors in this example), without shrinking the span:

$$span\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \underline{\boldsymbol{v}}_3\} = span\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2\}.$$

The reason for this is that

$$c_1\underline{\boldsymbol{v}}_1 + c_2\underline{\boldsymbol{v}}_2 + c_3\underline{\boldsymbol{v}}_3 = c_1\underline{\boldsymbol{v}}_1 + c_2\underline{\boldsymbol{v}}_2 + c_3\left(-3.5\,\boldsymbol{v}_1 + 1.5\,\underline{\boldsymbol{v}}_2\right) = d_1\underline{\boldsymbol{v}}_1 + d_2\underline{\boldsymbol{v}}_2\,.$$