Math 2270-002 Week 6 notes
We will not necessarily finish the material from a given day's notes on that day. We may also add or subtract some material as the week progresses, but these notes represent an in-depth outline of what we plan to cover. These notes cover material in 3.1-3.3, as well as some concepts review for our Friday midterm, in the Wednesday notes. Our exam on Friday includes this material.

Mon Sept 24
- 3.2 properties of determinants

Announcements: There will be an exam practice session on Thursday 12:55 – 2:20, room TBA.

Warm-up Exercise: Compute

$$\begin{vmatrix} 1 & 0 & -1 & 2 \\ 2 & 1 & 1 & 0 \\ 2 & 0 & 1 & 1 \\ -1 & 0 & -2 & 1 \end{vmatrix}$$

Hint: find a good row or column for the initial expansion

$$|A| = \sum_{i=1}^{4} a_{i2} C_{i2} = 0\cdot(-M_{12}) + 1 M_{22} - 0 M_{32} + 0 M_{42}$$

$$C_{i2} = (-1)^{i+2} M_{i2}$$

$$= 1 \cdot \begin{vmatrix} 1 & -1 & 2 \\ 2 & 1 & 1 \\ -1 & -2 & 1 \end{vmatrix}$$

$$= 1 \left(1 \cdot \begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix} - -1 \begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix} + 2 \begin{vmatrix} 2 & 1 \\ -1 & -2 \end{vmatrix} \right)$$

$$= 3 + 3 + 2(-3) = 0$$

today we'll see that the efficient way to compute determinants for larger matrices is to use facts about what elementary row operations do to determinants recall from Friday that

<u>Definition:</u> Let $A_{n \times n} = [a_{ij}]$. Then the determinant of A, written $det(A)$ or $|A|$, is defined by

$$det(A) := \sum_{j=1}^{n} a_{1j}(-1)^{1+j}M_{1j} = \sum_{j=1}^{n} a_{1j}C_{1j}.$$

Here M_{1j} is the determinant of the $(n-1) \times (n-1)$ matrix obtained from A by deleting the first row and the j^{th} column, and C_{1j} is simply $(-1)^{1+j}M_{1j}$.

More generally, the determinant of the $(n-1) \times (n-1)$ matrix obtained by deleting row i and column j from A is called the <u>ij Minor</u> M_{ij} of A, and $C_{ij} := (-1)^{i+j}M_{ij}$ is called the <u>ij Cofactor</u> of A.

<u>Theorem:</u> $det(A)$ can be computed by expanding across any row, say row i:

$$det(A) := \sum_{j=1}^{n} a_{ij}(-1)^{i+j}M_{ij} = \sum_{j=1}^{n} a_{ij}C_{ij}$$

or by expanding down any column, say column j:

$$det(A) := \sum_{i=1}^{n} a_{ij}(-1)^{i+j}M_{ij} = \sum_{i=1}^{n} a_{ij}C_{ij}.$$

(proof is not so easy - our text skips it and so will we. If you look on Wikipedia and as we illustrated for 3×3 matrices on Friday, the determinant is actually a sum of n factorial terms, each of which is $\pm$ a product of n entries of A where each product has exactly one entry from each row and column. The $\pm$ sign has to do with whether the corresponding permutation is even or odd. You can verify this pretty easily for the 2×2 and 3×3 cases. One can show, using math induction, that each row or column cofactor expansion above reproduces this sum, in the $n \times n$ case.)

We also illustrated the following theorem, which we will understand the reasons for partly today, and partly on Tuesday:

<u>Theorem</u> Let A be a square matrix. Then A^{-1} exists if an only if $|A| \neq 0$. And in this case

$$A^{-1} = \frac{1}{|A|} C^T$$

where $C = [C_{ij}]$ is the matrix of cofactors of A. (The matrix C^T is called the *"adjoint matrix"* in most texts, although ours prefers to use the word "*adjugate*", because "adjoint" has another meaning as well in linear algebra.)

Exercise 1) Compute the following determinants by being clever about which rows or columns to use:

1a) $\begin{vmatrix} 1 & 38 & 106 & 3 \\ 0 & 2 & 92 & -72 \\ 0 & 0 & 3 & 45 \\ 0 & 0 & 0 & -2 \end{vmatrix}$;

$= 1 \cdot \begin{vmatrix} 2 & 92 & -72 \\ 0 & 3 & 45 \\ 0 & 0 & -2 \end{vmatrix} - 0 + 0 - 0$

$= 1 \cdot \left(2 \begin{vmatrix} 3 & 45 \\ 0 & -2 \end{vmatrix} - 0 + 0 \right)$

$= 1 \cdot 2 \cdot \left(3 \underset{\substack{\uparrow \\ \det}}{|-2|} - 0 \right) = 1 \cdot 2 \cdot 3\,(-2)$

Theorem If A is upper Δ'ular then $|A|$ is the product of the diagonal entries

(pf: successively expand down first columns)

1b) $\begin{vmatrix} 1 & 0 & 0 & 0 \\ \pi^2 & 2 & 0 & 0 \\ 0.476 & 88 & 3 & 0 \\ 1 & 22 & 33 & -2 \end{vmatrix}$.

$= 1 \cdot \begin{vmatrix} 2 & 0 & 0 \\ 88 & 3 & 0 \\ 22 & 33 & -2 \end{vmatrix} - 0 + 0 - 0$

$= 1 \cdot \left(2 \begin{vmatrix} 3 & 0 \\ 33 & -2 \end{vmatrix} - 0 + 0 \right)$

$= 1 \cdot 2 \cdot 3 \cdot (-2)$

Theorem If A is lower Δ'ular det A is product of diagonal entries

Remark It's a Theorem that $\det A^T = \det A$. (One can check this by induction, after checking it for 1×1 & 2×2 dets)
So these theorems about upper and lower Δ'ular matrices are equivalent, although easy to check directly as we did.

Exercise 2) Explain why it is always true that for an upper triangular matrix (as in 1a), or for a lower triangular matrix (as in 1b), the determinant is always just the product of the diagonal entries.

The effective way to compute determinants for larger-sized matrices without lots of zeroes is to not use the definition directly, but rather to use elementary row operations to make the matrix upper triangular, along with the following facts, which track how elementary row operations affect determinants.

- (1a) Swapping any two rows of a matrix changes the sign of the determinant.
  proof: This is clear for 2×2 matrices, since

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12} \qquad \begin{vmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{vmatrix} = a_{21}a_{12} - a_{11}a_{22}$$

For 3×3 determinants, expand across the row *not* being swapped, and use the 2×2 swap property to deduce the result. Prove the general result by induction: once it's true for $n \times n$ matrices you can prove it for any $(n+1) \times (n+1)$ matrix, by expanding across a row that wasn't swapped, and applying the $n \times n$ result.

e.g., what happens when you swap row_1 & row_3 in 3×3 case.
Ans: expand across unswapped row_2:

from row_1

Original det: $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = -a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{22}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{23}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$

from row_3

If you swap row_1 & row_3 you also swap the rows in the 2×2 minor determinants, so you change the sign of each det, so you change the sign of 3×3 det.

Then proceed by induction as indicated.

(1b) Thus, if two rows in a matrix are the same, the determinant of the matrix must be zero: on the one hand, swapping those two rows leaves the matrix and its determinant unchanged; on the other hand, by (1a) the determinant changes its sign. The only way this is possible is if the determinant is zero.

Let A^* be the matrix with two identical rows of A swapped
Then $|A^*| = -|A|$
But $A^* = A$!
so $|A| = -|A| \implies 2|A| = 0$

• (2a) If you factor a constant out of a row, then you factor the same constant out of the determinant. Precisely, using $\mathscr{R}_i$ for i^{th} row of A, and writing $\mathscr{R}_i = c\,\mathscr{R}_i^*$

$$\begin{vmatrix} \mathscr{R}_1 \\ \mathscr{R}_2 \\ \vdots \\ \mathscr{R}_i \\ \mathscr{R}_n \end{vmatrix} = \begin{vmatrix} \mathscr{R}_1 \\ \mathscr{R}_2 \\ \vdots \\ c\,\mathscr{R}_i^* \\ \mathscr{R}_n \end{vmatrix} = c \begin{vmatrix} \mathscr{R}_1 \\ \mathscr{R}_2 \\ \vdots \\ \mathscr{R}_i^* \\ \mathscr{R}_n \end{vmatrix} .$$

proof: expand across the i^{th} row, noting that the corresponding cofactors don't change, since they're computed by deleting the i^{th} row to get the corresponding minors:

$$det(A) = \sum_{j=1}^{n} a_{ij} C_{ij} = \sum_{j=1}^{n} c\, a_{ij}^* C_{ij} = c \sum_{j=1}^{n} a_{ij}^* C_{ij} = c\, det\left(A^*\right) .$$

(2b) Combining (2a) with (1b), we see that if one row in A is a scalar multiple of another, then $det(A) = 0$.

- (3) If you replace row i of A, $\mathscr{R}_i$ by its sum with a multiple of another row, say $\mathscr{R}_k$ then the determinant is unchanged! Expand across the i^{th} row:

$$\begin{vmatrix} \mathscr{R}_1 \\ \mathscr{R}_2 \\ \mathscr{R}_k \\ \mathscr{R}_i + c\,\mathscr{R}_k \\ \mathscr{R}_n \end{vmatrix} = \sum_{j=1}^{n} \left(a_{ij} + c\,a_{kj}\right) C_{ij} = \sum_{j=1}^{n} a_{ij} C_{ij} + c \sum_{j=1}^{n} a_{kj} C_{ij} = det(A) + c \begin{vmatrix} \mathscr{R}_1 \\ \mathscr{R}_2 \\ \mathscr{R}_k \\ \mathscr{R}_k \\ \mathscr{R}_n \end{vmatrix} = det(A) + 0\,.$$

<u>Remark:</u> The analogous properties hold for corresponding "elementary column operations". In fact, the proofs are almost identical, except you use the corresponding column expansions instead of row expansions.

Exercise 3) Recompute $\begin{vmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{vmatrix}$ from Friday (using row and column expansions we always got an answer of 15 then.) This time use elementary row operations (and/or elementary column operations) to reduce the matrix into triangular form first.

Exercise 4) Compute $\begin{vmatrix} 1 & 0 & -1 & 2 \\ 2 & 1 & 1 & 0 \\ 2 & 0 & 1 & 1 \\ -1 & 0 & -2 & 1 \end{vmatrix}$.

We did this in warm-up.
Now we'll do it using elementary row operations.

$$= \begin{vmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 3 & -4 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & -3 & 3 \end{vmatrix} \quad \begin{matrix} \\ -2R_1+R_2 \to R_2 \\ -2R_1+R_3 \to R_3 \\ R_1+R_4 \to R_4 \end{matrix}$$

$$= \begin{vmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 3 & -4 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 0 \end{vmatrix} \quad R_3 + R_4 \to R_4$$

$= 0.$