Friday Sept 21

- 3.1 introduction to determinants

Announcements:

Warm-up Exercise: Look over the part of this week's HW having to do with affine transformations. (handout)

Try to construct the formula for the function which transforms "Bob" into medium-large Bob located at upper left of page for w5.2 problem

Math 2270-002
Homework due September 26.

Recall that problems which are not underlined are good for seeing if you can work with the underlying concepts; only the underlined problems need to be handed in. The Wednesday quiz will be drawn from all of these concepts and from these or related problems.

3.1 Introduction to determinants
1, **3**, **9**, 15, **25**, **27**, **29**, **31**, 32, 39, 40, 41

3.2: Properties of determinants
1, 2, 3, 4, 5, **21**, **25**, **27**, **29**, **31**, **33**, **39**

3.3: *Determinants and linear transformations; adjoint formula and Cramer's rule.*
3, 5, 13, **18**, **21**, **23**, **27**, 29, **31**

w5.1a) Use Cramer's rule to re-solve for x and y in the linear system **w4.1c** from previous homework, namely

$$\begin{bmatrix} 3 & 2 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 6 \end{bmatrix}.$$

w5.1b) Compute the determinants of the two matrices in **w4.2** from previous homework, and verify that the determinant test correctly identifies the invertible matrix. The two matrices were

$$A := \begin{bmatrix} -1 & 1 & -4 \\ -1 & -1 & 2 \\ 4 & 1 & 1 \end{bmatrix} \qquad B := \begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 3 \\ -2 & 1 & -2 \end{bmatrix}$$

w5.1c) Use the adjoint formula to re-find B^{-1} in **w5.1b**.

w5.1d) Use B^{-1} to solve the system

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 3 \\ -2 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$

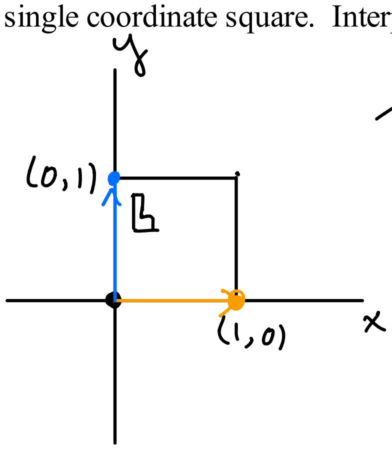
w5.1e) Re-solve for the y -variable in **w5.1d)**, using Cramer's Rule.

The following discussion and problems are related to section 1.9 and to our discussion of determinants in Chapter 3.

An *affine transformation* is a composition of a translation and a linear transformation. (When you talked about "tangent approximations" to functions in multivariable Calculus you were often talking about affine transformations. Single variable and multivariable differential calculus is built on the idea that for small scales, differentiable functions can be approximated well by affine functions.) In the following problems we'll specialize to affine transformations F from $\mathbb{R}^2$ to $\mathbb{R}^2$, i.e. functions of the form

$$F\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix}.$$

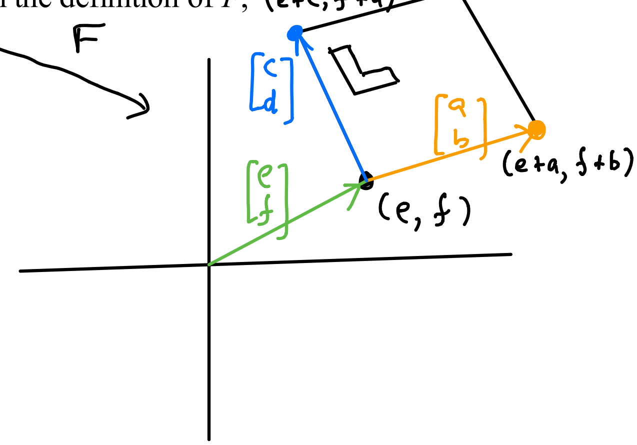
Since linear transformations transform families of parallel lines into families of parallel lines (or to points), the same is true for affine transformations. So as long as the transformations are 1-1 they will transform rectangular grid systems into parallelgram ones, and you can understand the entire grid from the image of a single coordinate square. Interpreting this algebraically, note that from the definition of F,



$$F\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} e \\ f \end{bmatrix}$$

$$F\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix}$$

$$F\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} c \\ d \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix},$$



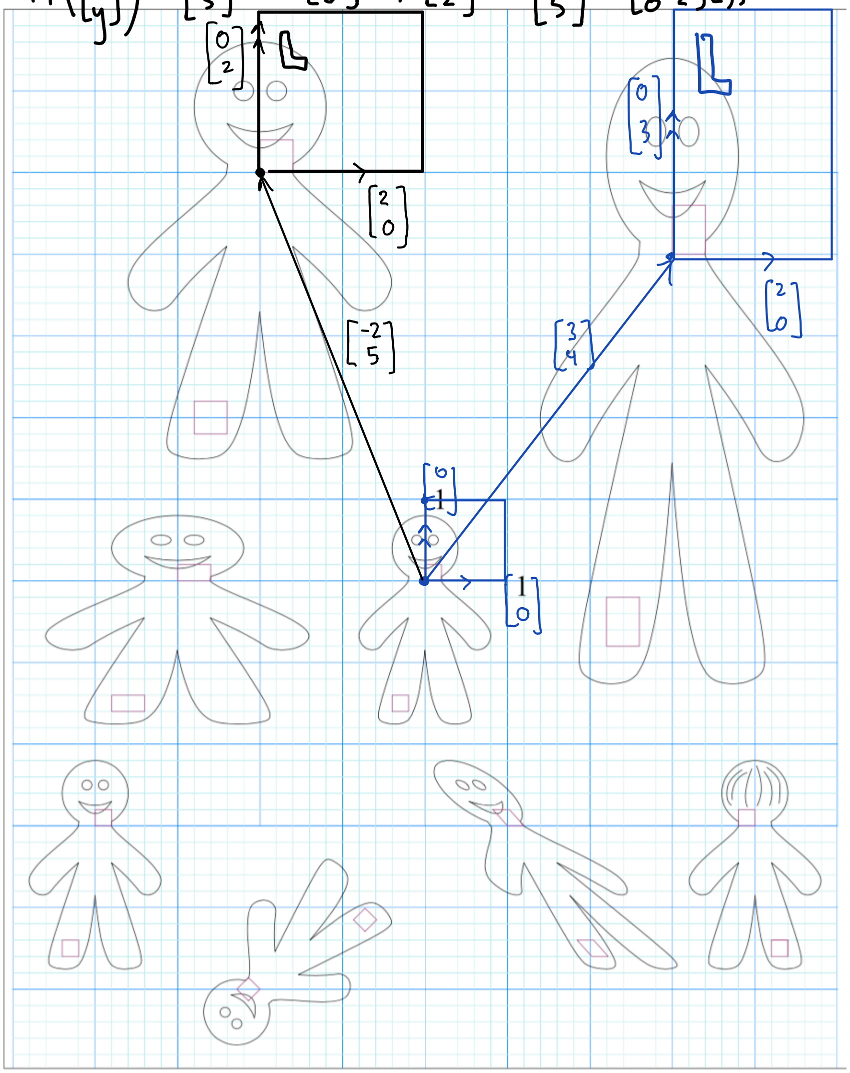
So you can reconstruct the translation vector and the two matrix columns for the affine function as soon as you know the images of $\underline{\mathbf{0}}, \underline{e}_1, \underline{e}_2$. For example, I reconstructed the transformation formula for Giant Bob in the upper right corner of the next page. Notice that Giant Bob on the next page has six times the area of original Bob - since original Bob can be filled up with different-sized squares, and the images of those squares will be rectangles having six times the original areas. There is an interesting connection between area expansion factors of affine transformations, and the determinants of the associated matrices. Recall that the determinant of a 2 by 2 matrix is given by

$$det\begin{bmatrix} a & c \\ b & d \end{bmatrix} = ad - bc.$$

Note that the determinant of Giant Bob's transformation matrix also equals 6.

warm-up

$$H\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 5 \end{bmatrix} + x\begin{bmatrix} 2 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$



$$G\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

(Think of it in this order:

$$G\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 4 \end{bmatrix} + x\begin{bmatrix} 2 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 3 \end{bmatrix}$$)

area expansion factor $= 6$

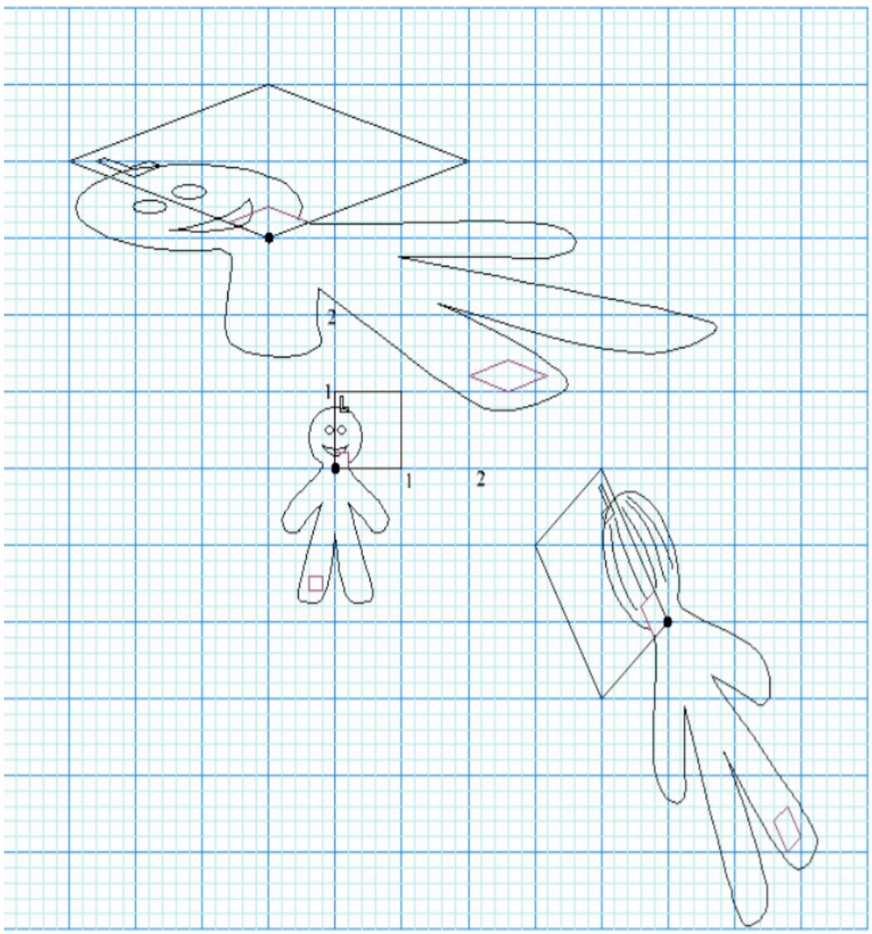
$$= \det\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

w5.2 Reconstruct the formulas for at least three more of the six (non-identity) transformations of Bob on the previous page, and comment on how the areas of the transformed Bobs are related to the determinants of the matrices in the transformations. Note that the Bob in the lower right corner got flipped over.

w5.3

a Find formulas for the two affine transformations of Bob indicated below.

b Squares in original Bob get transformed into parallelgrams in the image Bobs, and the area expansion factors are independent of the size of the original squares. So, you can deduce the area expansion factor for the image Bobs just by computing the area of the parallelgram image of the unit square. How do your area expansion factors in these two examples compare to the matrix determinants from the affine transformations?

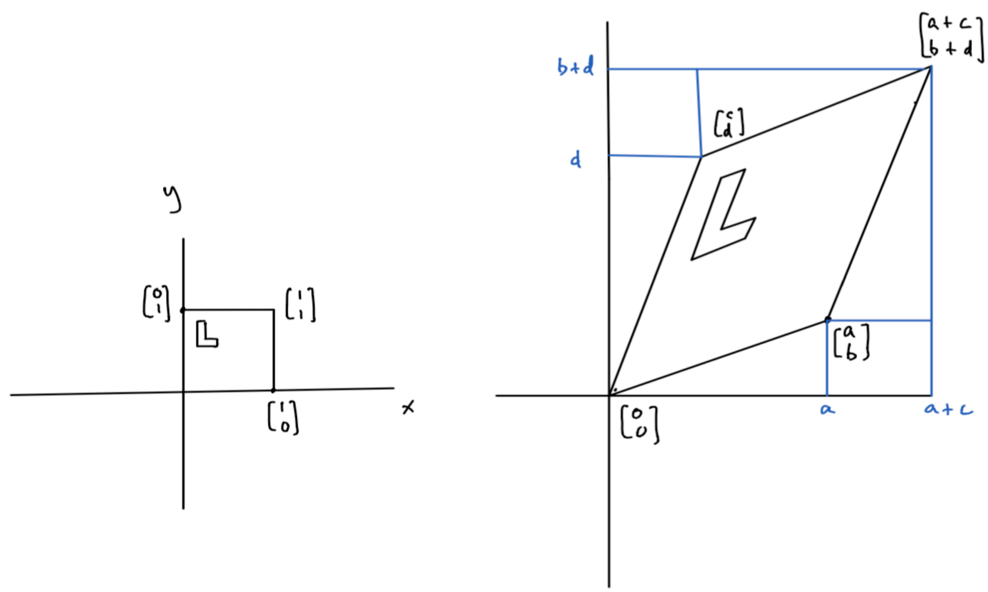


We'll talk more systematically about area/volume expansion factors and in arbitrary dimension, in class, but for affine transformations from $\mathbb{R}^2 \to \mathbb{R}^2$ one can use geometry to connect determinants to area expansion factors:

<u>w5.4</u> Can you compute the area of the parallelgram below (in terms of the letters a, b, c, d)? Since translations don't effect area, this will give the area expansion factor also for the images of arbitrary regions, under affine transformations that do include a translation term

$$F\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix}.$$

Hint: Start with the area of the large rectangle of length $a + c$ and height $b + d$, then subtract off the areas of the triangles and rectangles on the outside of the parallelgram. For convenience I chose the case where all of a, b, c, d are positive, and where the transformation didn't "flip" the parallelgram:



Determinants are scalars defined for square matrices $A_{n \times n}$ They always determine whether or not the inverse matrix A^{-1} exists, (i.e. whether the reduced row echelon form of A is the identity matrix): In fact, the determinant of A is non-zero if and only if A^{-1} exists. The determinant of a 1×1 matrix $[a_{11}]$ is defined to be the number a_{11}; determinants of 2×2 matrices are defined as in yesterday's notes; and in general determinants for $n \times n$ matrices are defined recursively, in terms of determinants of $(n-1) \times (n-1)$ submatrices:

Definition: Let $A_{n \times n} = [a_{ij}]$. Then the determinant of A, written $det(A)$ or $|A|$, is defined by

$$det(A) := \sum_{j=1}^{n} a_{1j}(-1)^{1+j} M_{1j} = \sum_{j=1}^{n} a_{1j} C_{1j}.$$

Here M_{1j} is the determinant of the $(n-1) \times (n-1)$ matrix obtained from A by deleting the first row and the j^{th} column, and C_{1j} is simply $(-1)^{1+j} M_{1j}$.

More generally, the determinant of the $(n-1) \times (n-1)$ matrix obtained by deleting row i and column j from A is called the $i\,j$ Minor M_{ij} of A, and $C_{ij} := (-1)^{i+j} M_{ij}$ is called the $i\,j$ Cofactor of A.

Exercise 1 Check that the messy looking definition above gives the same answer we talked about ~~yesterday~~ earlier this week in the 2×2 case, namely

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}.$$

$$= a_{11}(-1)^{1+1}M_{11} + a_{12}(-1)^{1+2}M_{12}$$

$$= a_{11}(1)a_{22} + a_{12}(-1)a_{21}$$

$$= a_{11}a_{22} - a_{21}a_{12} \quad \checkmark$$

from the last page, for our convenience:

<u>Definition:</u> Let $A_{n \times n} = [a_{ij}]$. Then the determinant of A, written $det(A)$ or $|A|$, is defined by

$$det(A) := \sum_{j=1}^{n} a_{1j}(-1)^{1+j}M_{1j} = \sum_{j=1}^{n} a_{1j}C_{1j}.$$

Here M_{1j} is the determinant of the $(n-1) \times (n-1)$ matrix obtained from A by deleting the first row and the j^{th} column, and C_{1j} is simply $(-1)^{1+j}M_{1j}$.

<u>Exercise 2</u> Work out the expanded formula for the determinant of a 3×3 matrix. It's not worth memorizing (as opposed to the recursive formula above), but it's good practice to write out at least once, and we might point to it later.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}(-1)^{1+1}M_{11} + a_{12}(-1)^{1+2}M_{12} + a_{13}(-1)^{1+3}M_{13}$$

$$= a_{11}(+1)\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12}(-1)\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{32} \end{vmatrix} + a_{13}(+1)\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{32} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

This expands to a sum of six terms. Let's organize them by whether they have + or − coefficients:

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$
$$- a_{11}a_{23}a_{32} - a_{12}a_{23}a_{31} - a_{13}a_{22}a_{31}$$

Interesting facts, which true for $n \times n$ determinants (see <u>Wikipedia</u>)

- each product contains exactly one entry from each row and column
- all such products are accounted for.
  (In the 3×3 case as you go down the rows there are 3 choices for the row 1 column, then two for the row 2 column, then one for the row 3 column, i.e. $3\cdot2\cdot1 = 3! = 6$ terms)
- the + or − sign depends on whether it takes an even or odd # of column interchanges to get their ordering back to $(1,2,3)$, when the products are written as above, with rows in 1-2-3 order. (This is called the "sign" of the column permutation.)

<u>Theorem:</u> $det(A)$ can be computed by expanding across any row, say row i:

$$det(A) := \sum_{j=1}^{n} a_{ij}(-1)^{i+j} M_{ij} = \sum_{j=1}^{n} a_{ij} C_{ij}$$

or by expanding down any column, say column j:

$$det(A) := \sum_{i=1}^{n} a_{ij}(-1)^{i+j} M_{ij} = \sum_{i=1}^{n} a_{ij} C_{ij}.$$

<u>Exercise 3a)</u> Let $A := \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{bmatrix}$. Compute $det(A)$ using the definition. (On the next page we'll use other rows and columns to do the computation.)

$$|A| = a_{11} M_{11} + a_{12}(-M_{12}) + a_{13} M_{13}$$

$$= 1 \begin{vmatrix} 3 & 1 \\ -2 & 1 \end{vmatrix} - 2 \begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} - 1 \begin{vmatrix} 0 & 3 \\ 2 & -2 \end{vmatrix}$$

$$= 1 \cdot 5 - 2(-2) - 1(-6)$$

$$= 5 + 4 + 6 = 15$$

From previous page,

$$A := \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{bmatrix}.$$

<u>3b)</u> Verify that the matrix of all the cofactors of A is given by $[C_{ij}] = \begin{bmatrix} 5 & 2 & -6 \\ 0 & 3 & 6 \\ 5 & -1 & 3 \end{bmatrix}$. Then expand $det(A)$ down various columns and rows using the a_{ij} factors and C_{ij} cofactors. Verify that you always get the same value for $det(A)$, as the Theorem on the previous page guarantees. Notice that in each case you are taking the dot product of a row (or column) of A with the corresponding row (or column) of the cofactor matrix.

$$[C_{ij}] = \begin{bmatrix} +\begin{vmatrix} 3 & 1 \\ -2 & 1 \end{vmatrix} & -\begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} & +\begin{vmatrix} 0 & 3 \\ 2 & -2 \end{vmatrix} \\ -\begin{vmatrix} 2 & -1 \\ -2 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} \\ +\begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 5 & 2 & -6 \\ 0 & 3 & -3 \\ 5 & -1 & 3 \end{bmatrix}$$

$$A := \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{bmatrix} \qquad [C_{ij}] = \begin{bmatrix} 5 & 2 & -6 \\ 0 & 3 & 6 \\ 5 & -1 & 3 \end{bmatrix}$$

$row_1(A) \cdot row_1(C) =$ (previous page) $5 + 4 + 6 = 15$

$row_2(A) \cdot row_2(C) = 0 + 9 + 6 = 15$

$col_2(A) \cdot col_2(C) = 4 + 9 + 2 = 15$

3c) What happens if you take dot products between a row of A and a *different* row of $[C_{ij}]$? A column of A and a *different* column of $[C_{ij}]$? The answer may seem magic. We'll come back to this example when we talk about the magic formula for the inverses of 3×3 (or $n \times n$) invertible matrices.

$$A := \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{bmatrix} \qquad [C_{ij}] = \begin{bmatrix} 5 & 2 & -6 \\ 0 & 3 & 6 \\ 5 & -1 & 3 \end{bmatrix}$$

$row_1(A) \cdot row_2(C) = 0 + 6 - 6 = 0$

$row_3(A) \cdot row_1(C) = 10 - 4 - 6 = 0$

$col_2(A) \cdot col_1(C) = 10 + 0 - 10 = 0$

So what does AC^T equal? Note, the columns of C^T are the rows of C, so we're recomputing the various row dot products

$$AC^T = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 & 5 \\ 2 & 3 & -1 \\ -6 & 6 & 3 \end{bmatrix} = \begin{bmatrix} 15 & 0 & 0 \\ 0 & 15 & 0 \\ 0 & 0 & 15 \end{bmatrix}$$

so $A^{-1} = \frac{1}{15} C^T = \frac{1}{|A|} C^T$ where C is the cofactor matrix.

this works for $n \times n$ matrices A!!!
(we'll see why next week)