There are situations where we are guaranteed a basis of $\mathbb{R}^n$ made out eigenvectors of A:

Theorem 1: Let A be an $n \times n$ matrix with distinct real eigenvalues λ_1, λ_2 , ... λ_n. Let $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$ be corresponding (non-zero) eigenvectors, $A\,\underline{\boldsymbol{v}}_j = \lambda_j\,\underline{\boldsymbol{v}}_j$. Then the set

$$\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n\}$$

is linearly independent, and so is a basis for $\mathbb{R}^n$.....this is one theorem we can prove!

I'll add proof if it's not in the text

example

Exercise 1 Monday: $A = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$

$p(\lambda) = (\lambda-4)(\lambda-1)$

$E_{\lambda=4} = \text{span}\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$

$E_{\lambda=1} = \text{span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$

note, we see $\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$ are a basis for $\mathbb{R}^2$

Theorem 1 ~~gua~~ guarantees this in general case

Exercise 2) Find the eigenvalues and eigenspace bases for the matrix below, and explain why there is no basis for $\mathbb{R}^2$ consisting of eigenvectors for this matrix: This was our warm-up exercise!

$$C = \begin{bmatrix} 3 & 2 \\ 0 & 3 \end{bmatrix}.$$

Theorem 2

Let $A_{n \times n}$ have factored characteristic polynomial

$$p(\lambda) = (-1)^n(\lambda - \lambda_1)^{k_1}(\lambda - \lambda_2)^{k_2} \ldots (\lambda - \lambda_m)^{k_m}$$

where like terms have been collected so that each λ_j is distinct (i.e different). Notice that

$$k_1 + k_2 + \ldots + k_m = n$$

because the degree of $p(\lambda)$ is n.

- Then $1 \leq dim\left(E_{\lambda=\lambda_j}\right) \leq k_j$. If $dim\left(E_{\lambda=\lambda_j}\right) < k_j$ then the λ_j eigenspace is called defective.
- The matrix A is diagonalizable if and only if each $dim\left(E_{\lambda=\lambda_j}\right) = k_j$. In this case, one obtains an $\mathbb{R}^n$

eigenbasis simply by combining bases for each eigenspace into one collection of n vectors. (The same definitions and reasoning can apply to complex eigenvalues and eigenvectors, and a basis of $\mathbb{C}^n$.)

(The proof of this theorem is fairly involved. It is illustrated in a positive way by Exercise 1, and in a negative way by Exercise 2.)

1 $B = \begin{bmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{bmatrix}$ $p(\lambda) = -(\lambda-2)^2(\lambda-3)$

dim = 2

$E_{\lambda=2} = span\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$

Theorem 2 good case

$E_{\lambda=3} = span\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

guarantees

$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

are a basis for $\mathbb{R}^3$

without the theorem we'd need to check $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

quick way: $det \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix} = 1(-1) - 0 + 1 \cdot 2$
$= 1 \neq 0$ ✓

2 $C = \begin{bmatrix} 3 & 2 \\ 0 & 3 \end{bmatrix}$ $p(\lambda) = (\lambda-3)^2$ bad

$E_{\lambda=3} = span\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ dim = 1

Theorem bad case.
(Tuesday warmup)

Wed Oct 31

- 5.3 diagonalizable matrices

Announcements: 5.3 HW 1, 3, 9, 11, 13, 21, 23, 25, 29, 31

more to come

(I hope to post the complete assignment, including a 10 pt. extra credit problem on Pac12 football rankings, by tonight.)

first finish Tues. notes

Warm-up Exercise: a) let $D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$. Find the eigenvalues and eigenvectors (eigenspace bases) for D.

b) What is the matrix power D^{10}?

Fact: for a diagonal matrix, the standard basis vectors are eigenvectors, the eigenvalues are corresponding diagonal entries.

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad ; \quad \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$E_{\lambda=2} = span\{\vec{e}_1\} \qquad E_{\lambda=3} = span\{\vec{e}_2\}.$$

don't need to use general algorithm.... but it would verify

(a) $|D - \lambda I| = \begin{vmatrix} 2-\lambda & 0 \\ 0 & 3-\lambda \end{vmatrix} = (2-\lambda)(3-\lambda) = (\lambda-2)(\lambda-3)$ evals $\lambda = 2, 3$ ✓

e.g. $E_{\lambda=2}$: $\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ backsolve $v_1 = t$ free, $v_2 = 0$

$Nul(D-2I)$

$\vec{v} = t\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

$E_{\lambda=2} = span\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$

(b) Recall, diagonal matrices are easy to multiply

e.g. $$\begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix}\begin{bmatrix} b_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & b_3 \end{bmatrix} = \begin{bmatrix} b_1a_1 & 0 & 0 \\ 0 & a_2b_2 & 0 \\ 0 & 0 & a_3b_3 \end{bmatrix}$$

(product of 2 diagonal matrices is diagonal; entries are the products of the corresponding entries in A, B)

$$D^{10} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}^{10} = \underbrace{\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\cdots\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}}_{10 \text{ times}} = \begin{bmatrix} 2^{10} & 0 \\ 0 & 3^{10} \end{bmatrix}$$

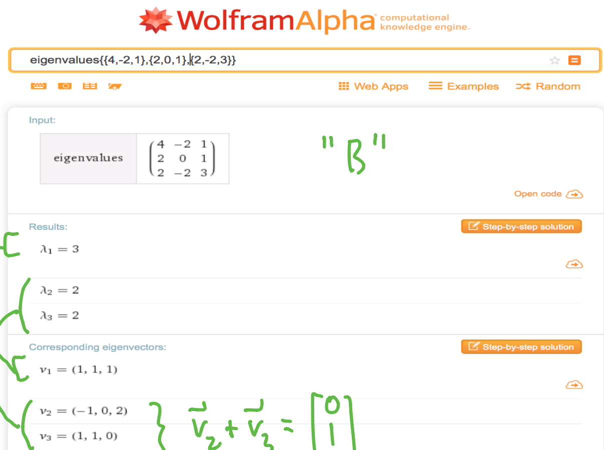
Continuing with the example from yesterday ...

If, for the matrix $A_{n \times n}$, there is a basis for $\mathbb{R}^n$ consisting of eigenvectors of A, then we can understand the geometry of the transformation

$$T(\underline{\mathbf{x}}) = A\,\underline{\mathbf{x}}$$

almost as well as if A is a diagonal matrix, and so we call such matrices *diagonalizable*. Having such a basis of eigenvectors for a given matrix is also extremely useful for algebraic computations, and will give another reason for the word *diagonalizable* to describe such matrices.

Use an $\mathbb{R}^3$ basis made of out eigenvectors of the matrix B in Exercise 1, yesterday, and put them into the columns of a matrix we will call P. We could order the eigenvectors however we want, but we'll put the $E_{\lambda=2}$ basis vectors in the first two columns, and the $E_{\lambda=3}$ basis vector in the third column:



"B"

$\lambda=3$

$\lambda=2$

$\vec{v}_2 + \vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$

$$\begin{array}{cccc} & \lambda=2 & \lambda=2 & \lambda=3 \\ P := & \multicolumn{3}{c}{\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & -2 & 1 \end{bmatrix}} \end{array}.$$

Now do algebra (check these steps and discuss what's going on!)

$$BP = \begin{bmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{bmatrix} \left[\begin{array}{c|c|c} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & -2 & 1 \end{array}\right]$$

$$= \left[\begin{array}{c|c|c} 0 & 2 & 3 \\ 2 & 0 & 3 \\ 4 & -4 & 3 \end{array}\right]$$

$\lambda=2 \quad \lambda=2 \quad \lambda=3$

$$= \left[\begin{array}{c|c|c} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & -2 & 1 \end{array}\right] \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

In other words,

$$B\,P = P\,D\,,$$

$B P P^{-1} = P D P^{-1}$

$B = P D P^{-1}$

$P^{-1} B P = P^{-1} P D$

$P^{-1} B P = D$

where D is the diagonal matrix of eigenvalues (for the corresponding columns of eigenvectors in P).
Equivalently (multiply on the right by P^{-1} or on the left by P^{-1}):

$$\boxed{B = P\,\mathrm{D}\,P^{-1}} \text{ and } P^{-1}BP = \mathrm{D}.$$

Exercise 1) Use one of the the identities above to show how B^{100} can be computed with only two matrix multiplications!

$$B^{100} = (PDP^{-1})(PDP^{-1})(PDP^{-1})\cdots(PDP^{-1})$$
$$= PD\underbrace{(P^{-1}P)}_{I}D\underbrace{(P^{-1}P)}_{I}D\cdots PDP^{-1}$$
$$= PD^{100}P^{-1}$$
$$B^{100} = P\begin{bmatrix} 2^{100} & 0 & 0 \\ 0 & 2^{100} & 0 \\ 0 & 0 & 3^{100} \end{bmatrix}P^{-1}$$

Definition: Let $A_{n \times n}$. If there is an $\mathbb{R}^n$ (or $\mathbb{C}^n$) basis $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, ..., \underline{\boldsymbol{v}}_n$ consisting of eigenvectors of A, then A is called diagonalizable. This is precisely why:

Write $A\,\underline{\boldsymbol{v}}_j = \lambda_j\,\underline{\boldsymbol{v}}_j$ (some of these λ_j may be the same, as in the previous example). Let P be the matrix

$$P = \left[\underline{\boldsymbol{v}}_1 \middle| \underline{\boldsymbol{v}}_2 \middle| \cdots \middle| \underline{\boldsymbol{v}}_n\right].$$

Then, using the various ways of understanding matrix multiplication, we see

$$A\,P = A\left[\underline{\boldsymbol{v}}_1 \middle| \underline{\boldsymbol{v}}_2 \middle| \cdots \middle| \underline{\boldsymbol{v}}_n\right] = \left[\lambda_1\underline{\boldsymbol{v}}_1 \middle| \lambda_2\underline{\boldsymbol{v}}_2 \middle| \cdots \middle| \lambda_n\underline{\boldsymbol{v}}_n\right]$$

$$= \left[\underline{\boldsymbol{v}}_1 \middle| \underline{\boldsymbol{v}}_2 \middle| \cdots \middle| \underline{\boldsymbol{v}}_n\right] \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ : & : & \ldots & : \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix}.$$

$$A\,P = P\,\mathrm{D}$$

$$A = P\,\mathrm{D}\,P^{-1}$$

$$P^{-1}A\,P = \mathrm{D}\,.$$

Unfortunately, as we've already seen, not all matrices are diagonalizable:

Exercise 2) Show that

$$C := \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

is not diagonalizable. (Even though it has the same characteristic polynomial as B, which was diagonalizable.