Exercise 2) a) Use the systematic algorithm to find the eigenvalues and eigenbases for the non-diagonal matrix of Exercise 2.

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}.$$

$A\vec{v} = \lambda\vec{v}.$

b) Use your work to describe the geometry of the linear transformation in terms of directions that get scaled:

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

① $(A-\lambda I)\vec{v} = \vec{0}$

$$|A-\lambda I| = 0 = \begin{vmatrix} 3-\lambda & 2 \\ 1 & 2-\lambda \end{vmatrix} = (3-\lambda)(2-\lambda) - 2 = \lambda^2 - 5\lambda + 6 - 2$$
$$= \lambda^2 - 5\lambda + 4$$
$$= (\lambda-4)(\lambda-1)$$
$$= 0 \text{ when } \lambda = 4, 1$$

$E_{\lambda=4} = Nul(A-4I)$

$$\left[\begin{array}{cc|c} -1 & 2 & 0 \\ 1 & -2 & 0 \end{array}\right] \quad \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$\boxed{E_{\lambda=4} = span\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}}$$

check. $\begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix} = 4\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ ✓

Tuesday

$E_{\lambda=1}$

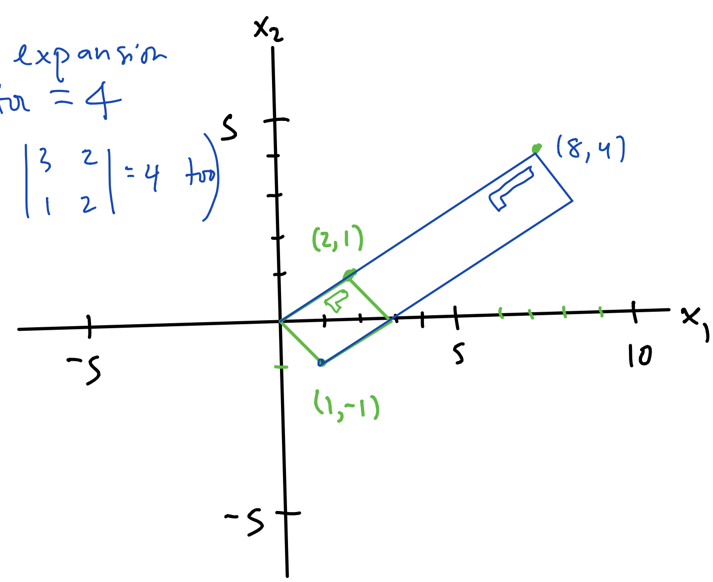
$$\left[\begin{array}{cc|c} 2 & 2 & 0 \\ 1 & 1 & 0 \end{array}\right] \quad \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$E_{\lambda=1} = span\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$$

1b) picture of $T(\vec{x}) = A\vec{x}$

area expansion factor = 4

(also $\begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix} = 4$ too)



Tues Oct 30

- 5.1-5.2 finding matrix eigenvalues and eigenvectors via the characteristic equation

Announcements: finish expl on Monday
then today's notes

$$\text{If} \quad A\vec{v} = \underset{\substack{\text{scalar}\\ \text{eigenvalue}}}{\lambda}\vec{v} \leftarrow \text{eigenvector}$$

Warm-up Exercise: Find eigenvalues, and then eigenspace bases for the matrix

$$C = \begin{bmatrix} 3 & 2 \\ 0 & 3 \end{bmatrix} = 3\begin{bmatrix} 1 & 2/3 \\ 0 & 1 \end{bmatrix} \quad \text{shear}$$

If

$$C\vec{v} = \lambda\vec{v}$$

$$C\vec{v} - \underbrace{\lambda\vec{v}}_{\lambda I} = \vec{0}$$

$$(C - \lambda I)\vec{v} = \vec{0}$$

$$\vec{v} \in Nul(C - \lambda I) \neq \{\vec{0}\}$$

so

$$\underbrace{|C - \lambda I|}_{\text{degree } n \text{ poly.}} = 0 \leftarrow \text{characteristic eqtn}$$

"characteristic polynomial"

$$C - \lambda I = \begin{bmatrix} 3 & 2 \\ 0 & 3 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

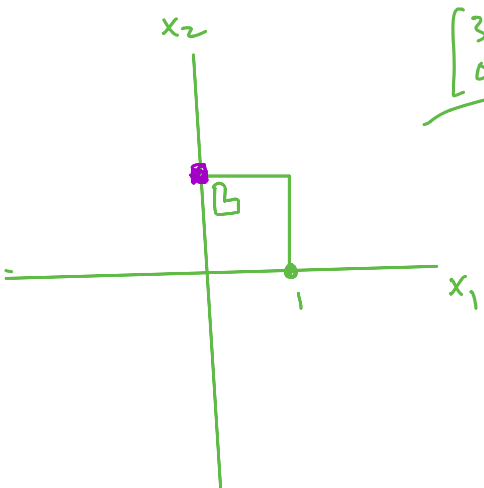
$$|C - \lambda I| = \begin{vmatrix} 3-\lambda & 2 \\ 0 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 \text{ set } = 0$$

$$\lambda = 3$$

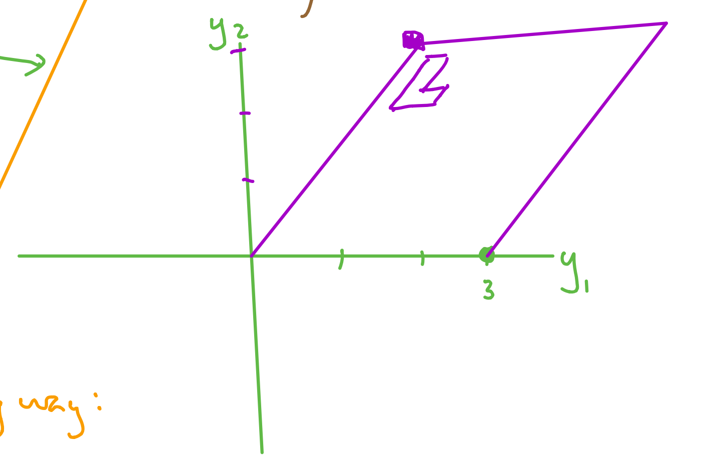
$E_{\lambda=3}$: $Nul(C - 3I)$

$$\left[\begin{array}{cc|c} 0 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$E_{\lambda=3} = span\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\} \quad \text{because } 1\cdot\begin{bmatrix} 0 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \vec{0}$$



$$\begin{bmatrix} 3 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$



long way:

$$\left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \text{backsolve} \quad \begin{array}{l} v_1 = t \text{ free} \\ v_2 = 0 \end{array} \quad \vec{v} = t\begin{bmatrix} 1 \\ 0 \end{bmatrix} \checkmark$$

Exercise 1) Find the eigenvalues and eigenspace bases for

$$B := \begin{bmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{bmatrix}.$$

(i) Find the characteristic polynomial and factor it to find the eigenvalues. $(p(\lambda) = -(\lambda - 2)^2(\lambda - 3))$

(ii) for each eigenvalue, find bases for the corresponding eigenspaces.

(iii) Can you describe the transformation $T(\underline{x}) = B\underline{x}$ geometrically using the eigenbases? Does $det(B)$ have anything to do with the geometry of this transformation?

(i) $|B-\lambda I| = 0$

$$\begin{vmatrix} 4-\lambda & -2 & 1 \\ 2 & -\lambda & 1 \\ 2 & -2 & 3-\lambda \end{vmatrix}$$

$$= \begin{vmatrix} 4-\lambda & -2 & 1 \\ 2 & -\lambda & 1 \\ 0 & \lambda-2 & 2-\lambda \end{vmatrix}$$

$-R_2+R_3 \to R_3$

$$= (2-\lambda)\begin{vmatrix} 4-\lambda & -2 & 1 \\ 2 & -\lambda & 1 \\ 0 & -1 & 1 \end{vmatrix}$$

$$= (2-\lambda)\begin{vmatrix} 4-\lambda & -1 & 1 \\ 2 & -\lambda+1 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

$col_3 + col_2 \to col_2$

$$= (2-\lambda)\begin{vmatrix} 4-\lambda & -1 \\ 2 & -\lambda+1 \end{vmatrix}$$

$$= (2-\lambda)\left[(4-\lambda)(-\lambda+1)+2\right]$$

$$= (2-\lambda)\left[\lambda^2 - 5\lambda + \underbrace{4+2}_{6}\right]$$

$$= (2-\lambda)\left[(\lambda-3)(\lambda-2)\right] = -(\lambda-2)^2(\lambda-3) = 0$$

$\lambda = 2, 3.$

$Nul(B-2I)$

$E_{\lambda=2}$:

$$\left[\begin{array}{ccc|c} 2 & -2 & 1 & 0 \\ 2 & -2 & 1 & 0 \\ 2 & -2 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & .5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$E_{\lambda=2} = span\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\2\end{bmatrix}\right\}$$

OR backsolve

$v_1 = v_2 - .5v_3$
$v_2 = v_2$ free
$v_3 = v_3$ free

$$\vec{v} = v_3\begin{bmatrix}-.5\\0\\1\end{bmatrix} + v_2\begin{bmatrix}1\\1\\0\end{bmatrix}$$

$$E_{\lambda=2} = span\left\{\begin{bmatrix}-1\\0\\2\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}\right\}$$

$E_{\lambda=3}$:

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 2 & -3 & 1 & 0 \\ 2 & -2 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 2 & -2 & \end{array}\right]$$

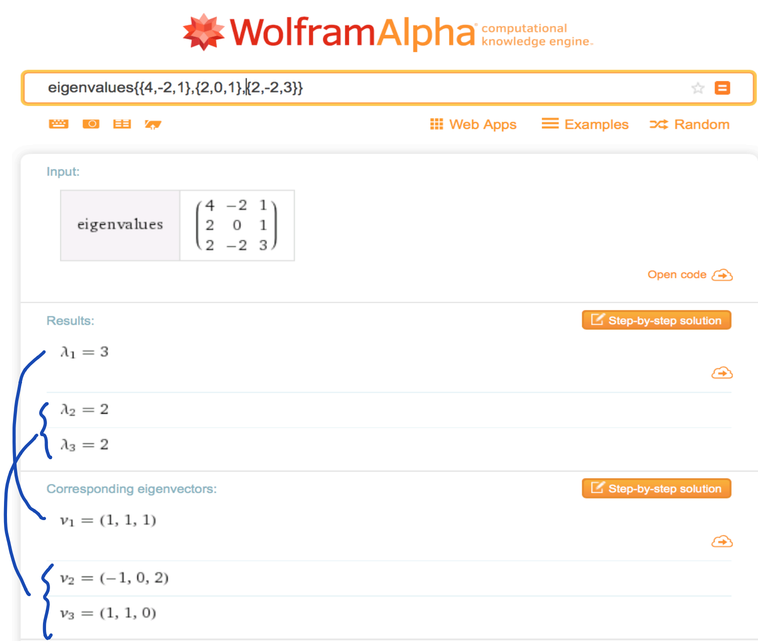
$-2R_1+R_2 \to R_2$
$-2R_1+R_3 \to R_3$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$2R_2+R_1 \to R_1$
$-2R_2+R_3 \to R_3$

$$E_{\lambda=3} = span\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$$

Your solution will be related to the output below:



It often turns out that by collecting bases from each eigenspace for the matrix $A_{n \times n}$, and putting them together, we get a basis for $\mathbb{R}^n$. This lets us understand the <u>geometry</u> of the transformation

$$T(\underline{\mathbf{x}}) = A\,\underline{\mathbf{x}}$$

almost as well as if A is a diagonal matrix. It does not always happen that the matrix A an basis of $\mathbb{R}^n$ made consisting of eigenvectors for A. (Even when all the eigenvalues are real.) When it does happen, we say that A is <u>diagonalizable.</u>

There are situations where we are guaranteed a basis of $\mathbb{R}^n$ made out eigenvectors of A:

Theorem 1: Let A be an $n \times n$ matrix with distinct real eigenvalues $\lambda_1, \lambda_2, \ldots. \lambda_n$. Let $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$ be corresponding (non-zero) eigenvectors, $A\,\underline{\boldsymbol{v}}_j = \lambda_j\,\underline{\boldsymbol{v}}_j$. Then the set

$$\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n\}$$

is linearly independent, and so is a basis for $\mathbb{R}^n$.....this is one theorem we can prove!

Exercise 2) Find the eigenvalues and eigenspace bases for the matrix below, and explain why there is no basis for $\mathbb{R}^2$ consisting of eigenvectors for this matrix: This was our warm-up exercise!

$$C = \begin{bmatrix} 3 & 2 \\ 0 & 3 \end{bmatrix}.$$