Part 2:
For an $m \times n$ matrix A there are actually four interesting subspaces. We've studied the first two of these quite a bit:

$$Nul\,A = \{\underline{\mathbf{x}} \in \mathbb{R}^n : A\,\underline{\mathbf{x}} = \underline{\mathbf{0}}\} \subseteq \mathbb{R}^n$$

and

$$Col\,A = \{\underline{\boldsymbol{b}} \in \mathbb{R}^m : \underline{\boldsymbol{b}} = A\,\underline{\mathbf{x}},\, \underline{\mathbf{x}} \in \mathbb{R}^n\} = span\{\underline{\boldsymbol{a}}_1, \underline{\boldsymbol{a}}_2, \ldots \underline{\boldsymbol{a}}_n\} \subseteq \mathbb{R}^m.$$

(Here we expressed $A = \left[\underline{\boldsymbol{a}}_1\ \underline{\boldsymbol{a}}_2\ \ldots\ \underline{\boldsymbol{a}}_n\right]$ in terms of its columns.) Through homework and class discussions we've understood the *rank+nullity Theorem*, that $dim\ Col\,A + dim\ Nul\,A = n$. This theorem follows from considerations of the reduced row echelon form of A and is connected to the number of pivot columns and the number of non-pivot columns in A. On Friday we introduced the other two of the four interesting subspaces connected to A. These are

$$Row\,A := span\{\underline{\boldsymbol{R}}_1, \underline{\boldsymbol{R}}_2, \ldots \underline{\boldsymbol{R}}_m\} \subseteq \mathbb{R}^n, \text{ where we write } A \text{ in terms of its rows, } A = \begin{bmatrix} \underline{\boldsymbol{R}}_1 \\ \underline{\boldsymbol{R}}_2 \\ \vdots \\ \underline{\boldsymbol{R}}_m \end{bmatrix}.$$

Note that $Row\,A = Col\,A^T$. The final subspace is

$$Nul\left(A^T\right) = \{\underline{\mathbf{y}} \in \mathbb{R}^m : A^T\,\underline{\mathbf{y}} = \underline{\mathbf{0}}\} \subseteq \mathbb{R}^m$$

<u>Exercise 1</u>: Using the reduced row echelon form of A we have realized the following facts. Let's review our reasoning, some of which you understood in last week's homework and all of which we've discussed in class. I've pasted our warm-up discussion from Friday into the following page, where we studied a large example in this context:

$$dim\,(Col\,A) = \#\ pivot\ columns\ in\ A\ (=\#pivots)$$

$$dim\,(Nul\,A) = \#\ non\text{-}pivot\ columns\ in\ A$$

$$dim\,(Row\,A) = \#\ pivot\ rows\ in\ A\ \ (=\#pivots)$$

$$dim\left(Nul\,A^T\right) = \#\ non\text{-}pivot\ columns\ in\ A^T.$$

The $dim\,(Col\,A) = dim\,(Row\,A)$ is called the *rank* of the matrix A and is the number of pivots in both A and A^T. If we call this number "r", then

$$dim\,(Nul\,A) = n - r$$
$$dim\left(Nul\,A^T\right) = m - r.$$

Example from Friday

Wednesday

basis for Col $A = \left\{ \begin{bmatrix}1\\2\\2\\1\end{bmatrix}, \begin{bmatrix}-1\\-3\\-2\\2\end{bmatrix}, \begin{bmatrix}5\\8\\2\\1\end{bmatrix} \right\}$

Math 2270-002
Friday October 19

Big example of four fundamental subspaces associated to each matrix.
Here is a matrix and it's reduced row echelon form, from quiz 7, and related matrices

basis for Nul A: $\left\{ \begin{bmatrix}-1\\-1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}-1\\-2\\0\\1\\0\end{bmatrix} \right\}$

$$A := \begin{bmatrix} 1 & -1 & 0 & -1 & 5 \\ 2 & -3 & -1 & -4 & 8 \\ 2 & -2 & 0 & -2 & 2 \\ 1 & 2 & 3 & 5 & 1 \end{bmatrix} \quad \text{row reduces to} \quad \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\vec{a}_1 \quad \vec{a}_2 \quad \vec{a}_3 \quad \vec{a}_4 \quad \vec{a}_5$

basis for Row A $= \left\{ \begin{bmatrix}1\\0\\1\\1\\0\end{bmatrix}^T, \begin{bmatrix}0\\1\\1\\2\\0\end{bmatrix}^T, \begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}^T \right\}$

$$A^T = \begin{bmatrix} 1 & 2 & 2 & 1 \\ -1 & -3 & -2 & 2 \\ 0 & -1 & 0 & 3 \\ -1 & -4 & -2 & 5 \\ 5 & 8 & 2 & 1 \end{bmatrix} \text{ row reduces to } \begin{bmatrix} 1 & 0 & 0 & \frac{9}{2} \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & \frac{5}{4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (A \text{ column reduces to } \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \frac{9}{2} & -3 & \frac{5}{4} & 0 & 0 \end{bmatrix}.)$$

What are $dim(Col\,A)$, $dim(Nul\,A)$, $dim(Row\,A)$, $dim(Nul\,A^T)$? Can you find bases for each subspace? How could you find these dimensions in general ?

dim (col A) = 3 = # of pivot cols in rref (A) because corresponding cols in A $(\vec{a}_1, \vec{a}_2, \vec{a}_5)$ are a pretty good basis for col A

dim (Nul A) = 2 = # of non-pivot cols
= # of free variables in solns $\vec{x}$ to $A\vec{x} = \vec{0}$
= # of basis vectors in Nul A

dim (row A) = 3 = # of pivot rows in rref (A) (same as # pivot cols, i.e # pivots)

dim (Nul A^T) = 1 = # of non-pivot cols in rref (A^T)

$T(x) = A\vec{x}$
$T: \mathbb{R}^5 \rightarrow \mathbb{R}^4$
Row A, Nul A $\subset \mathbb{R}^5$; Col A, Nul $A^T \subset \mathbb{R}^4$
Row A " col A^T

$S(\vec{y}) = A^T\vec{y}$
$S: \mathbb{R}^4 \rightarrow \mathbb{R}^5$

A is $m \times n$
$T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ $\quad T(x) = Ax$
$S: \mathbb{R}^m \rightarrow \mathbb{R}^n$ $\quad S(\vec{y}) = A^T\vec{y}$

We figured out that in general A has m rows & n colums

dim Col A = dim Row A this is called the rank of the matrix

dim Col A + dim Nul A = n (# of columns) (dim of domain $\mathbb{R}^n$)

dim Row A + dim Nul A^T = m (# of rows) (dim of codomain)

Geometry connected to the four fundamental subspaces:

- First, recall the geometry fact that the dot product of two vectors in $\mathbb{R}^n$ is zero if and only if the vectors are perpendicular, i.e.

$$\underline{\boldsymbol{u}} \cdot \underline{\boldsymbol{v}} = 0 \quad \text{if and only if } \underline{\boldsymbol{u}} \perp \underline{\boldsymbol{v}}.$$

(Well, we really only know this in $\mathbb{R}^2$ or $\mathbb{R}^3$ so far, from multivariable Calculus class. But it's true for all $\mathbb{R}^n$, as we'll see in Chapter 6.) So for a vector $\underline{\boldsymbol{x}} \in Nul\,A$ we can interpret the equation

$$A\,\underline{\mathbf{x}} = \underline{\mathbf{0}}$$

as saying that $\underline{\boldsymbol{x}}$ is perpendicular to every row of A. Because the dot product distributes over addition, we see that each $\underline{\boldsymbol{x}} \in Nul\,A$ is perpendicular to every linear combination of the rows of A, i.e. to all of $Row\,A$:

$$Row\,A \perp Nul\,A.$$

$$A\vec{x} = \vec{0}$$

$$= \underbrace{\begin{bmatrix} Row_1(A) \\ Row_2(A) \\ \vdots \\ Row_m(A) \end{bmatrix}}_{n \text{ cols}} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

i.e. $$\begin{bmatrix} Row_1(A)\cdot\vec{x} \\ Row_2(A)\cdot\vec{x} \\ \vdots \\ Row_m(A)\cdot\vec{x} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

And analogously,

$$Col\,A = Row\,A^T \perp Nul\,A^T$$

Wednesday warmup (in Monday notes)

small example.

$$T:\mathbb{R}^2 \to \mathbb{R}^2$$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

$$S\left(\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$$

1) check from Monday warmup:

$\text{Nul}\, A = \text{span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$ $\qquad$ $\text{Col}\, A = \text{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$

$\text{Row}\, A = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}^T\right\}$ $\qquad$ $\text{Nul}\, A^T = \text{span}\left\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right\}$

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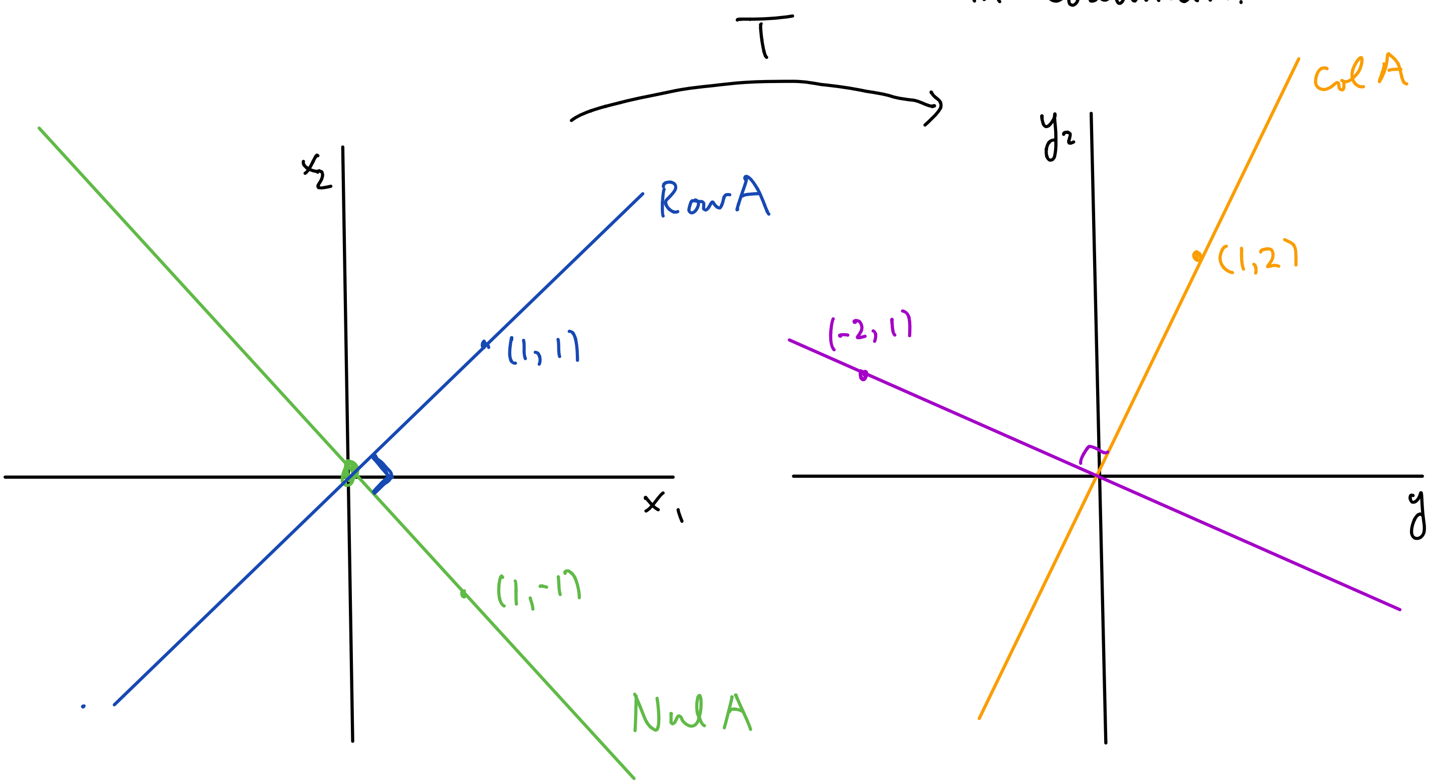
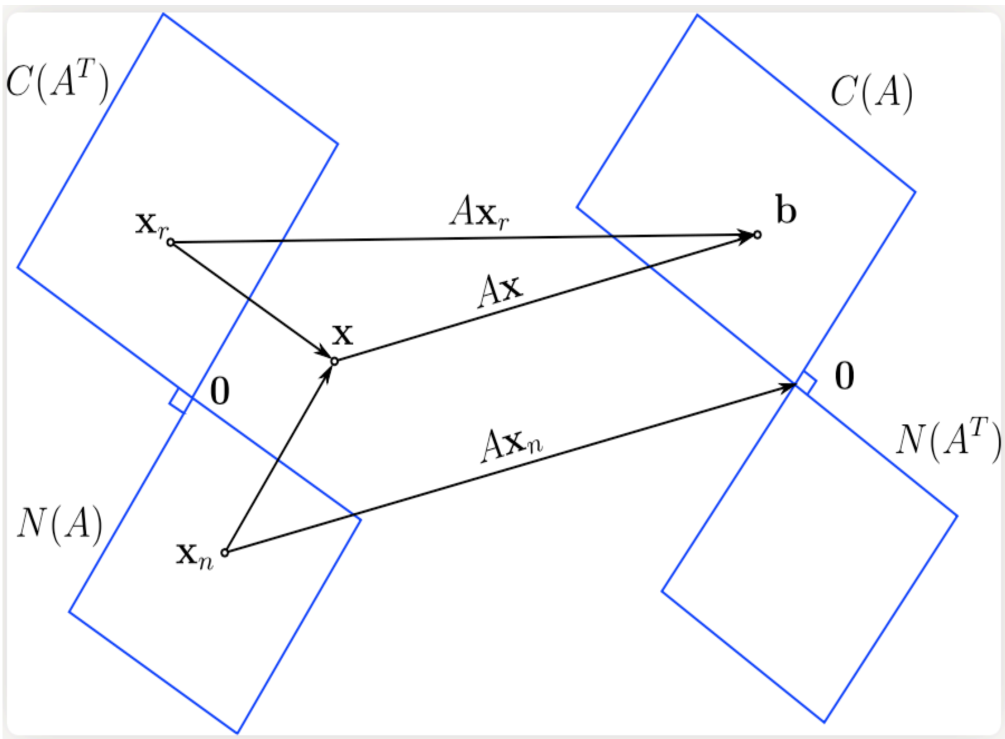
2) Sketch Nul A, Row A in domain $\qquad$ Sketch Col A, Nul A^T in codomain.



Here's a general schematic of what's going on, stolen from the internet. The web site I stole it from looks pretty good....

http://www.itshared.org/2015/06/the-four-fundamental-subspaces.html



More details on the decompositions .... we'll cover this in more detail in Chapter 6, but here's what true: In the domain $\mathbb{R}^n$, the two subspaces associated to A are *Row A* and *Nul A*. Notice that the only vector in their intersection is the zero vector, since

$$\underline{\boldsymbol{x}} \in Row\ A \cap Nul\ A \quad \Rightarrow \underline{\boldsymbol{x}} \cdot \underline{\boldsymbol{x}} = 0 \quad \Rightarrow \underline{\boldsymbol{x}} = \underline{\mathbf{0}}.$$

So, let

$$\{\underline{\boldsymbol{u}}_1, \underline{\boldsymbol{u}}_2, \ \dots \underline{\boldsymbol{u}}_r\} \qquad \text{be a basis for} \quad Row\ A$$

$$\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ \dots \underline{\boldsymbol{v}}_{n-r}\} \qquad \text{be a basis for } Nul\ A.$$

Then we can check that set of n vectors obtained by taking the union of the two sets,

$$\{\underline{\boldsymbol{u}}_1, \underline{\boldsymbol{u}}_2, \ \dots \underline{\boldsymbol{u}}_r, \underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ \dots \underline{\boldsymbol{v}}_{n-r}\}$$

is actually a basis for $\mathbb{R}^n$. This is because we can show that the n vectors in the set are linearly independent, so they automatically span $\mathbb{R}^n$ and are a basis: To check independence, let

$$c_1 \underline{\boldsymbol{u}}_1 + c_2 \underline{\boldsymbol{u}}_2 + \ \dots + c_r \underline{\boldsymbol{u}}_r + d_1 \underline{\boldsymbol{v}}_1 + d_2 \underline{\boldsymbol{v}}_2 + \ \dots + d_{n-r} \underline{\boldsymbol{v}}_{n-r} = \underline{\mathbf{0}}.$$

then

$$c_1 \underline{\boldsymbol{u}}_1 + c_2 \underline{\boldsymbol{u}}_2 + \ \dots + c_r \underline{\boldsymbol{u}}_r = -d_1 \underline{\boldsymbol{v}}_1 - d_2 \underline{\boldsymbol{v}}_2 - \ \dots - d_{n-r} \underline{\boldsymbol{v}}_{n-r}.$$

Since the vector on the left is in *Row A* and the one that it equals on the right is in *Nul A*, this vector is the zero vector:

$$c_1 \underline{\boldsymbol{u}}_1 + c_2 \underline{\boldsymbol{u}}_2 + \ \dots + c_r \underline{\boldsymbol{u}}_r = \underline{\mathbf{0}} = -d_1 \underline{\boldsymbol{v}}_1 - d_2 \underline{\boldsymbol{v}}_2 - \ \dots - d_{n-r} \underline{\boldsymbol{v}}_{n-r}.$$

Since $\{\underline{\boldsymbol{u}}_1, \underline{\boldsymbol{u}}_2, \ \dots \underline{\boldsymbol{u}}_r\}$ and $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ \dots \underline{\boldsymbol{v}}_{n-r}\}$ are each linearly independent sets, we deduce from these two equations that

$$c_1 = c_2 = \dots = c_r = 0, \qquad d_1 = d_2 = \dots = d_{n-r} = 0\ .$$

Q.E.D.

So the picture on the previous page is completely general (also for the decomposition of the codomain). One can check that the transformation $T(\underline{x}) = A\,\underline{\boldsymbol{x}}$ restricts to an isomorphism from *Row A* to *Col A*, because it is $1 - 1$ on these subspaces of equal dimension, so must also be onto. So, T squashes *Nul A*, and maps every translation of *Nul A* to a point in *Col A*. More precisely, Each

$$\underline{\boldsymbol{x}} \in \mathbb{R}^n$$

can be written uniquely as

$$\underline{\boldsymbol{x}} = \underline{\boldsymbol{u}} + \underline{\boldsymbol{v}} \qquad \text{with } \underline{\boldsymbol{u}} \in Row\ A,\ \underline{\boldsymbol{v}} \in Nul\ A.$$

and

$$T(\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}}) = T(\underline{\boldsymbol{u}}) + T(\underline{\boldsymbol{v}}) = T(\underline{\boldsymbol{u}}) \in Col\ (A).$$

As sets,

$$T(\{\underline{\boldsymbol{u}} + Nul\ A\}) = T(\underline{\boldsymbol{u}}).$$

Wed Oct 24

- 4.9 supplement: Google page rank

Announcements: We are where we are.

- today: start §4.9.
- Friday: finish 4.9 → google page rank

Warm-up Exercise: Do the "small example" of the geometry related to the 4 fundamental subspaces of a matrix, in Monday's notes