Math 2270-002 Week 9 notes
We will not necessarily finish the material from a given day's notes on that day. We may also add or subtract some material as the week progresses, but these notes represent an in-depth outline of what we plan to cover. These notes cover material in 4.5, 4.6, 4.9, 5.1-5.2.

Mon Oct 22
- 4.5, 4.6 Finish general theorems about finite dimensional vector spaces, bases, spanning sets, linearly independent sets and subspaces from 4.5; and complete the discussion of the four fundamental subspaces, from 4.6.

Announcements:

basis for subspace is set of ind. vectors that span the subspace

'til 12:58

Warm-up Exercise: For $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$

a) find a basis for Row A = span of rows of A

$\{[1 \quad 1]\}$

b) find a basis for Nul A = set of solutions $\vec{x}$ to $A\vec{x} = \vec{0}$

we know from rref(A) having one non-pivot column that dim Nul A = 1
homog soltns correspond to column dependencies, so...
basis for Nul A = $\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$

"long" way

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 2 & 2 & 0 \\ \hline 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

back solve: $x_1 = -x_2$, $x_2 = x_2$ free $\Rightarrow \vec{x} = x_2\begin{bmatrix} -1 \\ 1 \end{bmatrix} \Rightarrow$ basis $= \left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$ ✓

c) find a basis for Col A = span of cols of A

basis = $\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$

d) find a basis for Nul A^T

$A^T = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$

dim Nul A^T = 1

so basis = $\left\{\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right\}$.

Monday Review!

We've been studying *vector spaces*, which are a generalization of $\mathbb{R}^n$. They occur as *subspaces* of $\mathbb{R}^n$; also as vector spaces and subspaces of matrices, and of function spaces, for example. There are general theorems for vector spaces having to do with questions of *linear independence, span, basis, dimension* that we already understand well for $\mathbb{R}^n$. We ended Friday in the midst of a discussion of these theorems, and we'll complete that discussion in Part 1 of today's notes.

We've also been studying and using *linear transformations* $T : V \to W$ between vector spaces, which are generalizations of matrix transformations $T : \mathbb{R}^n \to \mathbb{R}^m$ given as $T(\underline{\boldsymbol{x}}) = A\,\underline{\boldsymbol{x}}$. A particularly useful linear transformation once if we have a basis $\beta = \{\underline{\boldsymbol{b}}_1, \underline{\boldsymbol{b}}_2, \ldots \underline{\boldsymbol{b}}_n\}$ for any vector space V is the coordinate transformation isomorphism:

$$\vec{v} = c_1\vec{b}_1 + c_2\vec{b}_2 + \cdots + c_n\vec{b}_n \in V$$

$$[\vec{v}]_\beta = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{R}^n$$

$$T(\underline{\boldsymbol{v}}) = [\underline{\boldsymbol{v}}]_\beta$$

$$T : V \to \mathbb{R}^n.$$

The coordinate transformation and its inverse function are helpful because they allow us to translate questions about linear independence and span in V into equivalent questions in $\mathbb{R}^n$, where we already have the tools to answer those questions.

For an $m \times n$ matrix A we've studied the subspaces $Nul\,A \subseteq \mathbb{R}^n$ and $Col\,A \subseteq \mathbb{R}^m$, which are the kernel and range of the associated linear transformation $T(\underline{\boldsymbol{x}}) = A\,\underline{\boldsymbol{x}}$, $T : \mathbb{R}^n \to \mathbb{R}^m$. On Friday we introduced two more subspaces connected to the geometry of the matrix transformations $T(\underline{\boldsymbol{x}}) = A\,\underline{\boldsymbol{x}}$. There are $Row\,A$ and $Nul\,A^T$. We'll complete the discussion of the four fundamental subspaces associated to matrix transformations in Part 2 of today's notes; we'll see how $Row\,A$ and $Nul\,A$ are related to a decomposition of the domain $\mathbb{R}^n$ of T, which is analogous to how $Col\,A = row(A^T)$ and $Nul\,A^T$ decompose the codomain $\mathbb{R}^m$.

on Friday:

Part I Monday
vector space theorems

There is a circle of ideas related to linear independence, span, and basis for vector spaces, which it is good to try and understand carefully. That's what we'll do today. These ideas generalize (and use) ideas we've already explored more concretely, and facts we already know to be true for the vector spaces $\mathbb{R}^n$. (A vector space that does not have a basis with a finite number of elements is said to be *infinite dimensional*. For example the space of all polynomials of arbitrarily high degree is an infinite dimensional vector space. We often study finite dimensional subspaces of infinite dimensional vector spaces.)

<u>Theorem 1</u> (constructing a basis from a spanning set): Let V be a vector space of dimension at least one, and let $span\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_p\} = V$.
Then a subset of the spanning set is a basis for V. (We followed a procedure like this to extract bases for *Col A*.)

here's how: If $\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_p\}$ is already independent, it's a basis and Theorem is proved.

otherwise, one of these vectors in a linear combo of the rest
by renumbering, assume

$$\vec{v}_p = d_1\vec{v}_1 + d_2\vec{v}_2 + \cdots + d_{p-1}\vec{v}_{p-1} \quad (\text{so } d_j \text{ may} = 0)$$

then any $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p$ is actually in $span\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_{p-1}\}$.
because

continue until the remaining set has same span as original set, but is now independent, so it's a basis

<u>Theorem 2</u> Let V be a vector space, with basis $\beta = \{\underline{\boldsymbol{b}}_1, \underline{\boldsymbol{b}}_2, \ldots \underline{\boldsymbol{b}}_n\}$. Then any set in V containing more than n elements must be linearly dependent. (We used reduced row echelon form to understand this in $\mathbb{R}^n$.)

Let $A = \{\vec{a}_1, \vec{a}_2, \ldots \vec{a}_N\}$ $N > n$ be a set in V.

Consider
$$c_1\vec{a}_1 + c_2\vec{a}_2 + \cdots + c_N\vec{a}_N = \vec{0}$$

and take coord transformation to $\mathbb{R}^n$, $[\ ]_\beta$

$$[\qquad '' \qquad]_\beta = [\vec{0}]_\beta = \vec{0} \in \mathbb{R}^n$$

vector eqtn in $\mathbb{R}^n$

$$c_1[\vec{a}_1]_\beta + c_2[\vec{a}_2]_\beta + \cdots + c_N[\vec{a}_N]_\beta = \vec{0}$$

$$n\left\{ \underbrace{\begin{bmatrix} [\vec{a}_1]_\beta & [\vec{a}_2]_\beta & \cdots & [\vec{a}_N]_\beta \end{bmatrix}}_{N} \right. \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{bmatrix} = \vec{0}$$

at least $N-n$ free parameters so lots of dependencies to be continued . . .

Theorem 3 Let V be a vector space, with basis $\beta = \{\underline{\boldsymbol{b}}_1, \underline{\boldsymbol{b}}_2, \ \ldots \ \underline{\boldsymbol{b}}_n\}$. Then no set $\alpha = \{\underline{\boldsymbol{a}}_1, \underline{\boldsymbol{a}}_2, \ \ldots \ \underline{\boldsymbol{a}}_p\}$ with $p < n$ vectors can span V. (We know this for $\mathbb{R}^n$.)

proof: If α did span V, then Theorem 1 tells us a subset of α is a basis for V.

It has $q \le p < n$ elements.

So by Theorem 2, since β has $n > q$ elements, β would be a dependent set but it's NOT, since it's a basis!

So α did not span all of V

$\blacksquare$

Theorem 4 Let V be a vector space, with basis $\beta = \{\underline{\boldsymbol{b}}_1, \underline{\boldsymbol{b}}_2, \ \ldots \ \underline{\boldsymbol{b}}_n\}$. Let $\alpha = \{\underline{\boldsymbol{a}}_1, \underline{\boldsymbol{a}}_2, \ \ldots \ \underline{\boldsymbol{a}}_p\}$ be a set of independent vectors that don't span V. Then $p < n$, and additional vectors can be added to the set α to create a basis $\{\underline{\boldsymbol{a}}_1, \underline{\boldsymbol{a}}_2, \ \ldots \ \underline{\boldsymbol{a}}_p, \ \ldots \ \underline{\boldsymbol{a}}_n\}$ (We followed a procedure like this when we figured out all the subspaces of $\mathbb{R}^3$.)

If α doesn't span all of V, pick any $\vec{a}_{p+1}$ not in span α (but in V)

Claim "new" $\alpha = \{\vec{a}_1, \vec{a}_2, \ldots \vec{a}_p, \vec{a}_{p+1}\}$ is still independent.

here's why:

dependency eqtn

$$* \quad c_1\vec{a}_1 + c_2\vec{a}_2 + \cdots + c_p\vec{a}_p + c_{p+1}\vec{a}_{p+1} = \vec{0}$$

case 1 if $c_{p+1} = 0$ then

$$c_1\vec{a}_1 + c_2\vec{a}_2 + \cdots + c_p\vec{a}_p = \vec{0}$$

so $c_1 = c_2 = \cdots = c_p = 0$ since "old" α is ind.

so also $p+1 \le n$ by Theorem 2 so $p < n$.

case 2 if $c_{p+1} \ne 0$ then from $*$

$$\vec{a}_{p+1} = \frac{c_1}{-c_{p+1}}\vec{a}_1 + \frac{c_2}{-c_{p+1}}\vec{a}_2 + \cdots + \frac{c_p}{-c_{p+1}}\vec{a}_p$$

case 2 can't happen by how we picked $\vec{a}_{p+1}$ NOT in span α

Continue until α has n vectors & is a basis

Theorem 5 Let Let V be a vector space, with basis $\beta = \{\underline{\boldsymbol{b}}_1, \underline{\boldsymbol{b}}_2, \ \dots \ \underline{\boldsymbol{b}}_n\}$. Then every basis for V has exactly n vectors. (We know this for $\mathbb{R}^n$.)

By Theorem 2, any set with more than n vectors in V is linearly dependent

By Theorem 3, any set with fewer than n vectors in V can't span.

So in order for a set to be a basis (ind & span) it must have exactly n vectors

Theorem 6 Let Let V be a vector space, with basis $\beta = \{\underline{\boldsymbol{b}}_1, \underline{\boldsymbol{b}}_2, \ \dots \ \underline{\boldsymbol{b}}_n\}$. If $\alpha = \{\underline{\boldsymbol{a}}_1, \underline{\boldsymbol{a}}_2, \ \dots \ \underline{\boldsymbol{a}}_n\}$ is another collection of exactly n vectors in V, and if $span\{\underline{\boldsymbol{a}}_1, \underline{\boldsymbol{a}}_2, \ \dots \ \underline{\boldsymbol{a}}_n\} = V$, then the set α is automatically linearly independent and a basis. Conversely, if the set $\{\underline{\boldsymbol{a}}_1, \underline{\boldsymbol{a}}_2, \ \dots \ \underline{\boldsymbol{a}}_n\}$ is linearly independent, then $span\{\underline{\boldsymbol{a}}_1, \underline{\boldsymbol{a}}_2, \ \dots \ \underline{\boldsymbol{a}}_n\} = V$ is guaranteed, and α is a basis. (We know all these facts for $\mathbb{R}^n$ from reduced row echelon form considerations.)

If <u>α spans V</u> and wasn't linearly independent then I could find subset of α that was smaller and a basis (Theorem 1), but Theorem 5 says all bases have the same # of vectors.

If α <u>is independent</u> but didn't span V, I could add vectors to it to get a basis with more elements than β but all bases have same # of elements, so α spans

Corollary Let Let V be a vector space of dimension n. Then the subspaces of V have dimensions $0, 1, 2, \ldots n-1, n$. (We know this for $\mathbb{R}^n$.)

Remark We used the coordinate transformation isomorphism between a vector space V with basis $\beta = \{\underline{\boldsymbol{b}}_1, \underline{\boldsymbol{b}}_2, \ldots \underline{\boldsymbol{b}}_n\}$ for Theorem 2, but argued more abstractly for the other theorems. An alternate (quicker) approach is to just note that because the coordinate transformation is an isomorphism it preserves sets of independent vectors, and maps spans of vectors to spans of the image vectors, so maps subspaces to subspaces. Then every one of the theorems above follows from their special cases in $\mathbb{R}^n$, which we've already proven. But this shortcut shortchanges the conceptual ideas to some extent, which is why we've discussed the proofs more abstractly.