There aren't very many sorts of subspaces (sub vector spaces) in $\mathbb{R}^n$.

Big Exercise: (We'll probably start this on Monday, and finish on Tuesday.) The vector space $\mathbb{R}^n$ has subspaces! But there aren't very many kinds, it turns out. (Even though there are countless kinds of *subsets* of $\mathbb{R}^n$.) Let's find *all* the possible kinds of subspaces of $\mathbb{R}^3$, using our expertise with matrix reduced row echelon form.

Subspaces W

(a) $\vec{0} \in W$

(b) $f, g \in W \Rightarrow f+g \in W$

(c) $f \in W, c \in \mathbb{R} \Rightarrow cf \in W$

implies

(d) $f, g \in W, c_1, c_2 \in \mathbb{R} \Rightarrow c_1 f + c_2 g \in W$ (use (c) then (d)).

(e) $f, g, h \in W, c_1, c_2, c_3 \in \mathbb{R} \Rightarrow (c_1 f + c_2 g) + c_3 h$ from (d), (b), (c)

...

collect subspaces from small to large in $\mathbb{R}^3$

Let W in $\mathbb{R}^3$ be a sub vector space

(0) W must contain $\vec{0}$

$\{\vec{0}\}$ is a subspace $\left(\begin{array}{l}\vec{0}+\vec{0}=\vec{0}\\ c\vec{0}=\vec{0}\end{array}\right)$

if W contains more,

let $\vec{u} \in W$, $\vec{u} \neq \vec{0}$.

(1) (c) $\Rightarrow$ span$\{\vec{u}\}$ contained W by (c).

Note span$\{\vec{u}\}$ is a subspace:

(b) $c_1\vec{u} + c_2\vec{u} = (c_1+c_2)\vec{u}$

(c) $c(c_1\vec{u}) = (cc_1)\vec{u}$

W could be span$\{\vec{u}\}$, i.e. a line thru $\vec{0}$.

(2) If W contains more than just span$\{\vec{u}\}$.

Let $\vec{v} \in W$, $\vec{v} \notin$ span$\{\vec{u}\}$

by (d) span$\{\vec{u}, \vec{v}\}$ is contained W

and span$\{\vec{u}, \vec{v}\}$ is a subspace:

(b) $(c_1\vec{u} + c_2\vec{v}) + (d_1\vec{u} + d_2\vec{v}) = (c_1+d_1)\vec{u} + (c_2+d_2)\vec{v}$

(c) $c(c_1\vec{u} + c_2\vec{v}) = cc_1\vec{u} + cc_2\vec{v}$

W could be a plane thru origin

$$\begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \\ u_3 & v_3 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & * \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\leftarrow$ can't happen because $\vec{v}$ is not multiple of $\vec{u}$

to be continued...

(3) If W contains more than span$\{\vec{u}, \vec{v}\}$

Let $\vec{w} \in W$, $\vec{w} \notin$ span$\{\vec{u}, \vec{v}\}$.

by (e), span$\{\vec{u}, \vec{v}, \vec{w}\}$ is in W

and span$\{\vec{u}, \vec{v}, \vec{w}\}$ is itself a subspace

b) $(c_1\vec{u} + c_2\vec{v} + c_3\vec{w}) + (d_1\vec{u} + d_2\vec{v} + d_3\vec{w}) = (c_1+d_1)\vec{u} + (c_2+d_2)\vec{v} + (c_3+d_3)\vec{w}$

(note, I'm not thinking about it, but actually using vector space axioms)

c) $c(c_1\vec{u} + c_2\vec{v} + c_3\vec{w}) = cc_1\vec{u} + cc_2\vec{v} + cc_3\vec{w}$

And what is span$\{\vec{u},\vec{v},\vec{w}\}$?

Has to be all of $\mathbb{R}^3$!

algebra!!

$$\begin{bmatrix} \vec{u} & \vec{v} & \vec{w} \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

means span$\{\vec{u},\vec{v},\vec{w}\} = \mathbb{R}^3$ (why?)

$$\begin{bmatrix} \vec{u} & \vec{v} & \vec{w} \end{bmatrix} \longrightarrow \cancel{\begin{bmatrix} 1 & 0 & z_1 \\ 0 & 1 & z_2 \\ 0 & 0 & 0 \end{bmatrix}}$$

would mean

$\vec{w} = z_1\vec{u} + z_2\vec{v}$

but $\vec{w}$ <u>not</u> in span$\{\vec{u},\vec{v}\}$.

Tues Oct 2

- 4.1-4.2 Vector spaces and subspaces; null spaces, column spaces.

Announcements: part of HW for 10/17

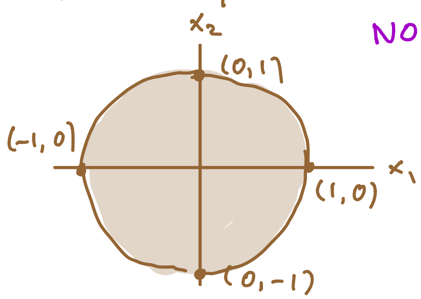
- 4.1 1, 3, 5, 7, 13, 19, 31 assigned yesterday
- 4.2 1, 3, 7, 9, 15, 17, 21, 25

Quiz tomorrow on material thru Wed

Finish big exercise from yesterday, continue with today's.

Warm-up Exercise: Which of these two sets is a subspace of $\mathbb{R}^2$. Why or why not?

a) $\{(x_1, x_2) \mid x_1^2 + x_2^2 \le 1\} = W_1$



NOT A SUBSPACE

a) TRUE

b) FALSE

just need one counterexample:

$\vec{u} = \begin{bmatrix}1\\0\end{bmatrix}, \vec{v} \in \begin{bmatrix}0\\1\end{bmatrix}$

are in W

$\vec{u} + \vec{v} = \begin{bmatrix}1\\1\end{bmatrix}$ not in W

(c) FALSE TOO

Recall, a subspace W is a sub vector space of the larger vector space ($\mathbb{R}^2$ in this case) so to be a subspace we need b), c) to hold:

a) $\vec{0} \in W$

b) $\vec{u}, \vec{v} \in W \Rightarrow \vec{u} + \vec{v} \in W$

c) $\vec{u} \in W,\ c \in \mathbb{R} \Rightarrow c\vec{u} \in W$

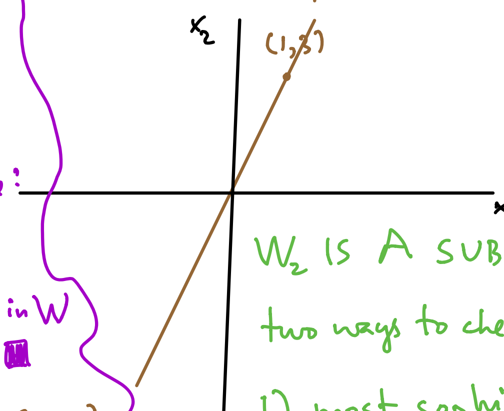
---

d) $\vec{u}, \vec{v} \in W,\ c_1, c_2 \in \mathbb{R}$

$\Rightarrow c_1\vec{u} + c_2\vec{v} \in W$

$\vdots$

b) $\{(x_1, x_2) \mid x_2 = 3x_1\} = W_2$



W_2 IS A SUBSPACE!

two ways to check

1) most sophisticated way:

write $W_2 = \text{span}\{\vec{u}\}$

$= \text{span}\left\{\begin{bmatrix}1\\3\end{bmatrix}\right\}$

$\left(\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}x_1\\3x_1\end{bmatrix} = x_1\begin{bmatrix}1\\3\end{bmatrix}\right)$

2) Or, verify (b), (c):

Let $\vec{u} = \begin{bmatrix}a_1\\3a_1\end{bmatrix}, \vec{v} = \begin{bmatrix}b_1\\3b_1\end{bmatrix}$ be in W_2

(b) $\vec{u} + \vec{v} \in W_2$?

$= \begin{bmatrix}a_1 + b_1\\3a_1 + 3b_1\end{bmatrix}$ satisfies $x_2 = 3x_1$ so $\vec{u} + \vec{v} \in W$

(c) $c\vec{u} = c\begin{bmatrix}a_1\\3a_1\end{bmatrix} = \begin{bmatrix}ca_1\\c3a_1\end{bmatrix}$ also satisfies $x_2 = 3x_1$

so $c\vec{u} \in W_2$

W_2 is a subspace

Finish the big exercise from Monday.

Review....
We've been discussing the abstract notions of *vector spaces* and *subspaces,* with some specific examples to help us with our intuition. Today we continue that discussion. We'll continue to use exactly the same language we used in Chapters 1-2 .... except now it's for general vector spaces:

Let V be a *vector space* (Do you recall that definition, at least roughly speaking?)

Definition: If we have a collection of p vectors $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_p\}$ in V, then any vector $\underline{\boldsymbol{v}} \in V$ that can be expressed as a sum of scalar multiples of these vectors is called a *linear combination* of them. In other words, if we can write

$$\underline{\boldsymbol{v}} = c_1\underline{\boldsymbol{v}}_1 + c_2\underline{\boldsymbol{v}}_2 + \ldots + c_p\underline{\boldsymbol{v}}_p ,$$

then $\underline{\boldsymbol{v}}$ is a *linear combination* of $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_p$. The scalars $c_1, c_2, \ldots, c_p$ are called the *linear combination coefficients* or *weights.*

Definition The *span* of a collection of vectors, written as *span*$\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_p\}$, is the collection of all linear combinations of those vectors.

Definition:
a) An indexed set of vectors $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_p\}$ in V is said to be *linearly independent* if no one of the vectors is a linear combination of (some) of the other vectors. The concise way to say this is that the only way $\underline{\mathbf{0}}$ can be expressed as a linear combination of these vectors,

$$c_1\underline{\boldsymbol{v}}_1 + c_2\underline{\boldsymbol{v}}_2 + \ldots + c_p\underline{\boldsymbol{v}}_p = \underline{\mathbf{0}} ,$$

is for all of the weights $c_1 = c_2 = \ldots = c_p = 0$.

b) An indexed set of vectors $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_p\}$ is said to be *linearly dependent* if at least one of these vectors is a linear combination of (some) of the other vectors. The concise way to say this is that there *is* some way to write $\underline{\mathbf{0}}$ as a linear combination of these vectors

$$c_1\underline{\boldsymbol{v}}_1 + c_2\underline{\boldsymbol{v}}_2 + \ldots + c_p\underline{\boldsymbol{v}}_p = \underline{\mathbf{0}}$$

where *not all* of the $c_j = 0$. (We call such an equation a *linear dependency.* Note that if we have any such linear dependency, then any $\underline{\boldsymbol{v}}_j$ with $c_j \neq 0$ is a linear combination of the remaining $\underline{\boldsymbol{v}}_k$ with $k \neq j$. We say that such a $\underline{\boldsymbol{v}}_j$ is *linearly dependent* on the remaining $\underline{\boldsymbol{v}}_k$.)

And from yesterday,

Definition: A *subspace* of a vector space V is a subset H of V which is itself a vector space with respect to the addition and scalar multiplication in V. As soon as one verifies a), b), c) below for H, it will be a subspace.

a) The zero vector of V is in H

b) H is closed under vector addition, i.e. for each $\underline{\boldsymbol{u}} \in H, \underline{\boldsymbol{v}} \in H$ then $\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}} \in H$.

c) H is closed under scalar multiplication, i.e for each $\underline{\boldsymbol{u}} \in H$, $c \in \mathbb{R}$, then also $c\,\underline{\boldsymbol{u}} \in H$.

Theorem (spans are subspaces) Let V be a vector space, and let $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots\ \underline{\boldsymbol{v}}_n\}$ be a set of vectors in V. Then $H = span\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots\ \underline{\boldsymbol{v}}_n\}$ is a subspace of V.
proof: We need to check that for $H = span\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots\ \underline{\boldsymbol{v}}_n\}$

a) The zero vector of V is in H

e.g. $\underset{\substack{\uparrow \\ H}}{0\vec{v}_1} = \vec{0}$ by (13)

b) H is closed under vector addition, i.e. for each $\underline{\boldsymbol{u}} \in H, \underline{\boldsymbol{v}} \in H$ then $\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}} \in H$.

Let $\vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$
$\vec{v} = d_1\vec{v}_1 + d_2\vec{v}_2 + \cdots + d_n\vec{v}$

Then $\vec{u} + \vec{v} \underset{\substack{\uparrow \\ \text{after many steps} \\ \text{using axioms } 1,2,3,6,\ldots}}{=\!=} (c_1+d_1)\vec{v}_1 + (c_2+d_2)\vec{v}_2 + \cdots + (c_n+d_n)\vec{v}_n.$

c) H is closed under scalar multiplication, i.e for each $\underline{\boldsymbol{u}} \in H$, $c \in \mathbb{R}$, then also $c\,\underline{\boldsymbol{u}} \in H$.

Let $\vec{u}$ as above

$c\vec{u} \underset{\text{after many steps}}{=\!=} cc_1\vec{v}_1 + cc_2\vec{v}_2 + \cdots + cc_n\vec{v}_n.$

Definition A *basis* of a vector space V is a set of vectors $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n\}$ in V which *spans* V and which is *linearly independent*.

(e.g. the standard basis, or other bases in $\mathbb{R}^n$, which we've discussed before.)

Definition A vector space V is *finite dimensional* if it has a basis with only a finite number of vectors in it. Otherwise V is called *infinite dimensional.*

Definition If V is a finite dimensional vectorspace, then the *dimension* of V is the number of vectors in a basis for V. (It turns out that every basis has the same number of vectors, just like every basis of $\mathbb{R}^n$ always has exactly n vectors.

more than n vectors in $\mathbb{R}^n$ are dependent

fewer than n vectors in $\mathbb{R}^n$ can't span $\mathbb{R}^n$.

Remark Using minimal spanning sets, i.e. bases, was how we were able to characterize all possible subspace of $\mathbb{R}^3$ yesterday (or today, if we didn't finish on Monday). Can you characterize all possible ~~subsets~~ subspaces of $\mathbb{R}^n$ in this way?

(0) $\{\vec{0}\}$ has dimension 0

(1) $\text{span}\{\vec{u}\}$ $\vec{u} \neq \vec{0}$ line thru origin dimension $= 1$

(2) plane thru origin, i.e. $\text{span}\{\vec{u}, \vec{v}\}$ $\vec{u}, \vec{v}$ ind. $\dim = 2$

(3) $\dim(\mathbb{R}^3) = 3$

Example: Let P_n be the space of polynomials of degree at most n,

$$P_n = \{p(t) = a_0 + a_1 t + a_2 t^2 + \dots + a_n t^n \textit{ such that } a_0, a_1, \dots a_n \in \mathbb{R}\}$$

Note that P_n is the span of the $(n+1)$ functions

$$p_0(t) = 1, p_1(t) = t, p_2(t) = t^2, \dots \; p_n(t) = t^n.$$

Although we often consider P_n as a vector space on its own, we can also consider it to be a subspace of the much larger vector space V of all functions from $\mathbb{R}$ to $\mathbb{R}$.

Exercise 1 abbreviating the functions by their formulas, we have

$$P_3 = span\{\ 1, t, t^2, t^3\ \}.$$

Are the functions in the set $\{\ 1, t, t^2, t^3\ \}$ linearly independent or linearly dependent? Are they a basis for P_3 ?

- By construction these functions span P_3.
- Think if you can show they're linearly independent

to be continued...

$$c_1 \cdot 1 + c_2 t + c_3 t^2 + c_4 t^3 = 0$$ ← the zero fcn, which is 0 for all t

$$\Longrightarrow c_1 = c_2 = c_3 = c_4 = 0?$$