Wed Nov 7

- Appendix B, the Complex plane $\mathbb{C}$. Brief review for exam 2.

- Substitute Notes Today !!

Announcements:
- If you made a print out for the Pac 12 problem, you can hand it in (otherwise you can upload the .m file to CANVAS, by 5:00 today)
- I'll post practice exam later today
- review session Thursday 1-2:20 LCB 222 (small chance this moves. If so, I'll let you know.)
- Friday exam 12:50-1:50 ↖ 5 minutes early to 5 minutes late.

Warm-up Exercise: no warm-up today.

Complex number algebra and geometry.
Appendix B of text
November 7

$$\mathbb{C} := \{a + b\,i \mid a, b \in \mathbb{R} \text{ and } i^2 := -1\}$$

Arithmetic: If $z = a + b\,i$ and $w = c + d\,i$ then

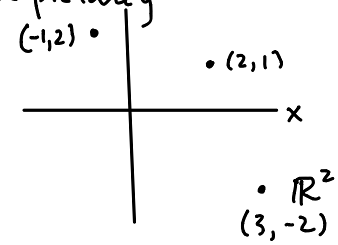
$$z = w \text{ if and only if } a = c \text{ and } b = d.$$
$$z + w := (a + c) + (b + d)\,i$$
$$(a+bi)(c+di) = \quad z\,w := (ac - bd) + (ad + bc)\,i$$

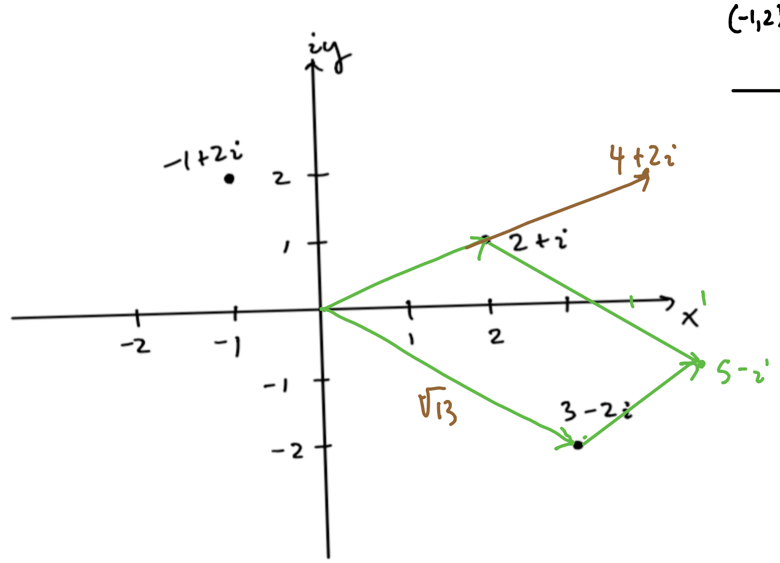
Focusing on just addition and real-number scalar multiplication, $\mathbb{C}$ can be thought of as a *real* vector space of dimension 2. In this case, the natural basis is $\beta = \{1, i\}$. Then the coordinate transfromation is an isomorphism with $\mathbb{R}^2$:

$$z = a + b\,i, \quad [z]_\beta = \begin{bmatrix} a \\ b \end{bmatrix}.$$

If we identify complex numbers with these coordinates in $\mathbb{R}^2$, we get the "complex plane" representation of $\mathbb{C}$:

coord picture





Exercise 1a) Illustrate that complex number addition corresponds to vector addition in the complex plane, i.e. in the $\mathbb{R}^2$ coordinate plane that we have identified with $\mathbb{C}$ as above. Use some of the points labeled above. Also, that real scalar multiplication corresponds to scalar multiplication in the $\mathbb{R}^2$ coordinate plane.

1b) We define the modulus of $z = a + b\,i$ to be $|z| = \sqrt{a^2 + b^2}$. Note that this is just the magnitude of the coordinate vector $[a, b]^T$. Compute the modulus of some of the vectors in the diagram above.

1a) $(2+i) + (3-2i) = 5 - i$ in $\mathbb{C}$ } "equivalent"

$\begin{bmatrix}2\\1\end{bmatrix} + \begin{bmatrix}3\\-2\end{bmatrix} = \begin{bmatrix}5\\-1\end{bmatrix}$ using β coords.

$2(2+i) = 4 + 2i \rightsquigarrow 2\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}4\\2\end{bmatrix}$

2b) $|3-2i| = \sqrt{9+4} = \sqrt{13}$

Interesting geometry starts happening when you combine the geometry of the complex plane~~s~~ with algebraic operations such as complex multiplication.

<u>Exercise 1</u> Define the transformation $T : \mathbb{C} \to \mathbb{C}$ by the formula

$$T(z) := iz \quad \text{i.e. } T(x+iy) := i(x+iy) = -y + ix.$$

<u>1a)</u> Verify that this is a linear transformation of $\mathbb{C}$.

(1) $T(z+w) = i(z+w) = iz + iw = T(z) + T(w)$

(2) c real: $T(cz) = icz = c(iz) = cT(z)$

<u>1b)</u> Describe T geometrically, in terms of its effect in the (x, y) coordinate plane. Include the matrix for T with respect to the basis $\beta = \{1, i\}$.

coord plane

$T(z) = iz$

$\vec{b}_1 = 1$

$T(1) = i \cdot 1 = i$

$[T(1)]_B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

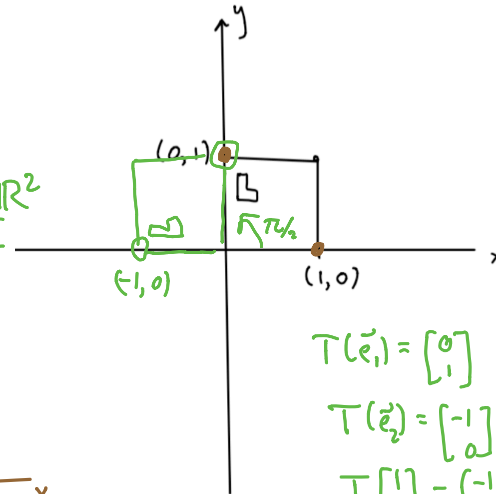
$\vec{b}_2 = i$

$T(i) = i \cdot i = -1$

$[T(i)]_B = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$$[T]_B = \left[[T(\vec{b}_1)]_B \ \ [T(\vec{b}_2)]_B \right] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

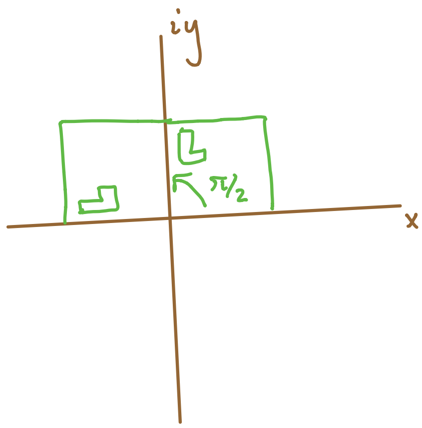
← acts as a rotation matrix in $\mathbb{R}^2$ same in $\mathbb{C}$



$T(\vec{e}_1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$T(\vec{e}_2) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$T \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$



Exercise 2 Let $a, b \in \mathbb{R}$. Define the linear transformation $T : \mathbb{C} \to \mathbb{C}$ by the formula

$$T(z) = (a + b\,i)z, \quad \text{i.e. } T(x + i\,y) = (a + b\,i)\cdot(x + i\,y).$$

2a)

Verify that this is a linear transformation of $\mathbb{C}$.

"Same" as in 1

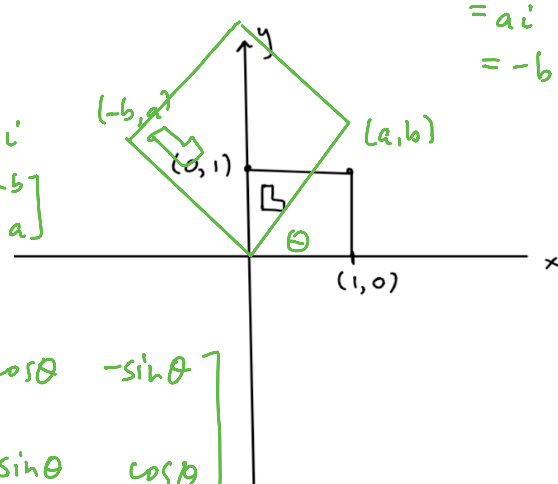
2b) Describe T geometrically, in terms of its effect in the (x, y) coordinate plane. Include the matrix for T with respect to the basis $\beta = \{1, i\}$.

Describe T geometrically, in terms of its effect in the (x, y) coordinate plane which we have identified with $\mathbb{C}$. Include the matrix for T with respect to the basis $\beta = \{1, i\}$. It should look familiar.

$(a+bi)i = ai + bi^2$
$= ai - b$
$= -b + ai$

$$[T]_\beta = \left[[T(\vec{b}_1)]_\beta \;\; [T(\vec{b}_2)]_\beta \right]$$

$T(z) = (a+bi)z$ | $T(1) = a + bi$ $\quad [a+bi]_\beta = \begin{bmatrix} a \\ b \end{bmatrix}$ $\qquad T(i) = -b + ai$ $\quad [-b+ai]_\beta = \begin{bmatrix} -b \\ a \end{bmatrix}$



$$[T]_B = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \sqrt{a^2+b^2}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

rotation-dilation

An important algebraic operation for complex numbers is *conjugation*:

Definition: Let $z = x + i\,y$. Then the *conjugate* of z, $\overline{z} := x - i\,y$. Geometrically this is a reflection in the complex plane, across the x-axis. But the major uses of conjugations are algebraic:

<u>Exercise 3</u> Let $z = x + i\,y$, $w = u + i\,v$ be complex numbers. Then

<u>3a)</u> $|z|^2 = z\,\overline{z}$.

<u>3b)</u> $\overline{z\,w} = \overline{z}\,\overline{w}$.

<u>3c)</u> $z\,w = 0$ if and only if $z = 0$ or $w = 0$.

<u>3d)</u> If $z \neq 0$ then $\frac{1}{z}$ exists (i.e. the multiplicative inverse), in fact, $\frac{1}{z} = \frac{\overline{z}}{|z|^2}$.

$\left(\frac{1}{z}\,\frac{\overline{z}}{\overline{z}} = \frac{\overline{z}}{|z|^2} \right)$

Geometric meaning of complex multiplication:

We use the *polar form* of complex numbers, which corresponds to polar coordinates in the $\mathbb{R}^2$ coordinate plane.

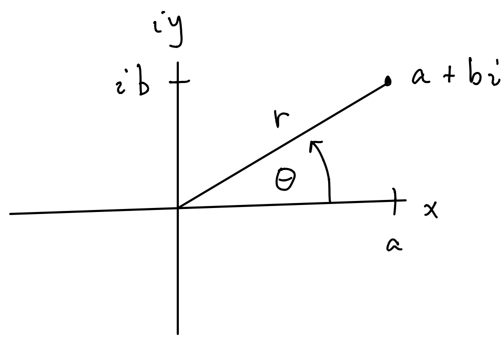
Let $z = a + b\,i$.

Let $r = \sqrt{a^2 + b^2} = |\,z\,|$.

Then

$$z = r\left(\frac{a}{\sqrt{a^2+b^2}} + \frac{b}{\sqrt{a^2+b^2}}\,i\right)$$
$$z = r\,(\cos\theta + i\sin\theta)$$

where θ is the polar coordinate angle.



Multiplication!! If

$$z = a + b\,i = r\,(\cos\theta + i\sin\theta)$$
$$w = c + d\,i = \rho\,(\cos\varphi + i\sin\varphi)$$

Then

$$z\,w = r\,(\cos\theta + i\sin\theta)\rho\,(\cos\varphi + i\sin\varphi)$$

$$z\,w = r\,\rho\,[(\cos\theta\cos\varphi - \sin\theta\sin\varphi) + i\,(\cos\theta\sin\varphi + \sin\theta\cos\varphi)]$$

$$z\,w = r\,\rho\,[\cos(\theta+\varphi) + i\sin(\theta+\varphi)].$$

upshot: when you multiply two complex numbers, their moduli are multiplied, and their polar angles are added!

Remark: Using *Euler's formula* that $e^{i\theta} = \cos\theta + i\sin\theta$ the computation above may be expressed as:
If

$$z = r\,e^{i\theta} \text{ and } w = \rho\,e^{i\varphi}$$

then

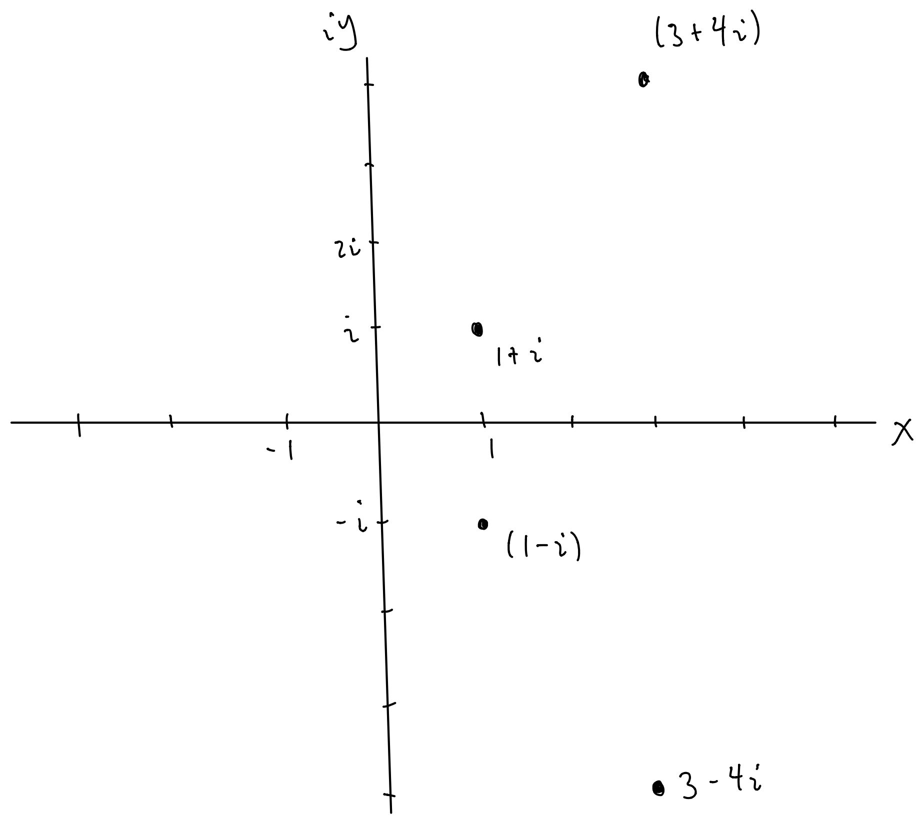
$$z\,w = r\,e^{i\theta}\,\rho\,e^{i\varphi} = r\,\rho\,e^{i\theta}e^{i\varphi} = r\,\rho\,e^{i(\theta+\varphi)}\,.$$

Exercise 4. Play with complex multiplication algebraically (using the rectangular coordinates of complex numbers) and geometrically (using their polar forms and the previous page).

$(3 - 4\,i) \cdot (3 + 4\,i) =$

$(1 + i)^2 =$

$(1 + i)^4 =$



Topics/concepts list for exam 2

4.1 vector spaces and sub vector spaces (subspaces) - abstract definitions.
realization of subspaces as null spaces or as spans of collections of vectors
how to check if a subset is a subspace.
examples such as polynomial vector spaces, matrix vector spaces, $\mathbb{R}^n$, and subspaces of all of these.

4.2 *Nul A* and *Col A* for $T(\underline{\mathbf{x}}) = A\,\underline{\mathbf{x}}$; *Kernel T* and *Range T* for general linear transformations $T : V \rightarrow W$
how to find *Nul A* and *Col A*, and bases for each.

4.3 linearly independent/dependent sets; bases for vector spaces (including subspaces).
how to check whether the vectors in a set span a vector space.
how to check whether a set of vectors is linearly independent.
how to build up bases as growing sets of independent vectors, one vector at a time, until the set spans.
how to cull dependent vectors from a spanning set, until it is an independent set.

4.4 every basis of n vectors for a vector space V yields a coordinate system, via the coordinate isomorphism with $\mathbb{R}^n$.
answering questions about span and linear independence for sets of vectors in V by using coordinates with respect to a basis.
favorite examples include P_n, $M_{m \times n}$, the polynomial and matrix spaces.

4.5 dimension of a vector space. basic facts about dimension, number of vectors required to span, maximum number of independent vectors, dimensions of subspaces.

4.6 rank of a matrix. rank + nullity theorem.
connection to reduced row echelon form of the matrix.
how to find *Row A*, *Nul* A^T.
what *Nul A*, *Row A*, *Col A*, *Nul* A^T have to do with the geometry of the transformation $T(\underline{\mathbf{x}}) = A\,\underline{\mathbf{x}}$.

4.9 Markov chains, stochastic and regular stochastic matrices, steady-state vector, google page rank ideas.

5.1-5.2 eigenvalues and eigenvectors. Finding eigenvalues via the characteristic equation $det(A - \lambda I) = 0$; finding eigenspace bases.

5.3 Diagonalizable and non-diagonalizable matrices. Algebraic consequences, e.g. computing large powers of diagonalizable matrices.

5.4 Matrices of linear transformations, given domain and codomain bases; change of basis for matrix transformations using "better bases".
Improved understanding of the transformation $T(\underline{\mathbf{x}}) = A\,\underline{\mathbf{x}}$ in terms of $\mathbb{R}^n$ basis made out of eigenvectors, as compared to the standard basis.

5.5 Complex eigendata. Finding complex eigenvalues and eigenvectors, especially for 2×2 matrices; rotation-dilation matrices.

computations
fluency in the defintions and concepts
ability to create examples illustrating definitions and concepts
ability to discern whether statements are true or false, based on the material we've covered.
use of material from Chapters 1-3 that relates to Chapters 4-5.