On Tuesday

5.5 Complex eigenvalues and eigenvectors.

We'll focus on 2 × 2 matrices, for simplicity. In this case it will turn out that a matrix with real entries and complex eigenvalues is always similar to a rotation-dilation matrix...

Definition A matrix of the form $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ is called a *rotation-dilation* matrix, because for $r = \sqrt{a^2 + b^2}$ we can rewrite A as

scale factor ↓ Rot$_\theta$

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \sqrt{a^2+b^2}\begin{bmatrix} \frac{a}{\sqrt{a^2+b^2}} & \frac{-b}{\sqrt{a^2+b^2}} \\ \frac{b}{\sqrt{a^2+b^2}} & \frac{a}{\sqrt{a^2+b^2}} \end{bmatrix}$$

$$A = r\begin{bmatrix} \frac{a}{r} & -\frac{b}{r} \\ \frac{b}{r} & \frac{a}{r} \end{bmatrix} = r\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

So the transformation $T(\underline{\boldsymbol{x}}) = A\,\underline{\boldsymbol{x}}$ rotates vectors by an angle θ and then scales them by a factor of r. (So A^2 rotates by an angle 2 θ and scales by r^2; A^3 rotates by an angle 3 θ and scales by r^3, etc.

Exercise 2) Draw the transformation picture for

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

A ↓

$$= \sqrt{2}\begin{bmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix}$$

$$A = \sqrt{2}\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

and interpret this transformation as a rotation-dilation. Overlay your diagram onto one picture: How about the transformation picture for $T \circ T = T^2$?

$T(\vec{e}_1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

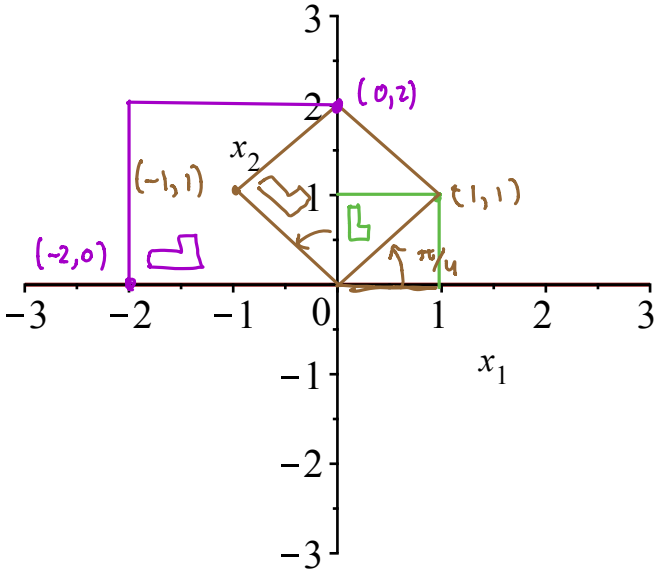
$T(\vec{e}_2) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$A^2 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$

$A^2\vec{e}_1 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

$A^2\vec{e}_2 = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$



Exercise 3) What are the eigenvalues of a rotation-dilation matrix $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$? How do you think you would go about finding the eigenvectors?

$\lambda = a \pm ib$

warmup.

follow the same algorithms.

Tues Nov 6

- 5.5 Complex eigendata

Announcements:

- On Wed, I won't start Chptr 6. Instead, Appendix on complex #'s in text plus tie in to things like "matrix of a linear transformation" ℂ is your friend
- I'll let you know when I have the review classroom (Th 1:00 – 2:20)
- I'll say something about midterm tomorrow.

Warm-up Exercise: Find the eigenvalues of this matrix, in terms of a, b:

$$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \qquad \lambda = a \pm bi$$

(Important matrix for §5.5)

$$|A - \lambda I| = \begin{vmatrix} a-\lambda & -b \\ b & a-\lambda \end{vmatrix} = (a-\lambda)^2 + b^2$$

$$= (\lambda - a)^2 + b^2$$

$$= (\lambda - a)^2 - (ib)^2 \qquad i^2 = -1$$

$$= (\lambda - a - ib)(\lambda - a + ib)$$

$$= [\lambda - (a+ib)][\lambda - (a-ib)]$$

$$\lambda = a \pm ib.$$

OR

$$(\lambda - a)^2 + b^2 \overset{\text{set}=0}{=} 0$$

$$(\lambda - a)^2 = -b^2$$

$$\lambda - a = \pm ib$$

$$\lambda = a \pm ib$$

It is possible for a matrix A with real entries to be diagonalizable if one allows complex scalars and vectors, even if it's not diagonalizable with real eigenvalues and eigenvectors. ~~You saw an example of that on a food for thought problem, if you weren't afraid.~~ We'll use a matrix today that we'll use later as well, in section 5.6, to study an interesting discrete dynamical system. This matrix is not a rotation-dilation matrix, but it is *similar* to one, and that fact will help us understand the discrete dynamical system.

Exercise 1) Let

$$B = \begin{bmatrix} .9 & -.4 \\ .1 & .9 \end{bmatrix}$$

Find the (complex) eigenvalues and eigenvectors for B.

$\text{Nul}(B-\lambda I)$ to be non-trivial

$\Leftrightarrow$

$\text{rref}(B-\lambda I) \neq I$

$\Leftrightarrow$

$\det(B-\lambda I) = 0.$

$(.9-\lambda)(.9-\lambda)$

① $|B-\lambda I| = \begin{vmatrix} .9-\lambda & -.4 \\ .1 & .9-\lambda \end{vmatrix} = (\lambda-.9)(\lambda-.9) + .04 = 0$

$(\lambda-.9)^2 = -.04 = (.2i)^2$

$\lambda - .9 = \pm .2i$

$\lambda = .9 \pm .2i$

(quicker than quadratic formula)

$E_{\lambda=.9+.2i}$

$\begin{pmatrix} .9-\lambda \\ = .9-(.9+.2i) \\ = -.2i \end{pmatrix}$

	$-.2i$	$-.4$	0
	$.1$	$-.2i$	0
$10R_2 \to R_1$	1	$-2i$	0
$5R_1 \to R_2$	$-i$	-2	0
	1	$-2i$	0
$iR_1 + R_2 \to R_2$	0	0	0

$i(-2i) - 2 = 2 - 2 = 0$

$E_{\lambda=.9+.2i} = \text{span}\left\{\begin{bmatrix} 2i \\ 1 \end{bmatrix}\right\}$

all complex multiples

$= \text{span}\left\{\begin{bmatrix} -2 \\ i \end{bmatrix}\right\}$

$E_{\lambda=.9-.2i}$

$\begin{pmatrix} .9-\lambda \\ = .9-(.9-.2i) \\ = +.2i \end{pmatrix}$

	$.2i$	$-.4$	0
	$.1$	$.2i$	0
$10R_2 \to R_1$	1	$2i$	0
$5R_1 \to R_2$	i	-2	0
	1	$2i$	0
$-iR_1 + R_2$	0	0	0

$E_{\lambda=.9-.2i} = \text{span}\left\{\begin{bmatrix} -2i \\ 1 \end{bmatrix}\right\}$

on the right side we just replaced all i's with $-i$'s

General facts we saw illustrated in the example, about complex eigenvalues and eigenvectors: Let A be a matrix with real entries, and let

$$A\,\underline{\boldsymbol{v}} = \lambda\,\underline{\boldsymbol{v}}$$

with $\lambda = a + b\,i$, $\underline{\boldsymbol{v}} = \underline{\boldsymbol{u}} + i\,\underline{\boldsymbol{w}}$ complex, $(a, b \in \mathbb{R}, \underline{\boldsymbol{u}}, \underline{\boldsymbol{w}} \in \mathbb{R}^n)$. Then we write

Re "real part"
Im "imag. part"

$$\text{Re}\,\lambda = a, \quad \text{Im}\,\lambda = b$$

$$\text{Re}\,\underline{\boldsymbol{v}} = \underline{\boldsymbol{u}}, \quad \text{Im}\,\underline{\boldsymbol{v}} = \underline{\boldsymbol{w}}.$$

must =

must =

So, the equation $A\,\underline{\boldsymbol{v}} = \lambda\,\underline{\boldsymbol{v}}$ expands as

$$A\vec{u} + iA\vec{w} = (a\vec{u} - b\vec{w}) + i(b\vec{u} + a\vec{w})$$

$$A\,(\underline{\boldsymbol{u}} + i\,\underline{\boldsymbol{w}}) = (a + b\,i)\,(\underline{\boldsymbol{u}} + i\,\underline{\boldsymbol{w}}).$$

It will always be true then that the conjugate $\lambda = a - b\,i$ is also an eigenvalue, and the conjugate vector $\underline{\boldsymbol{v}} = \underline{\boldsymbol{u}} - i\,\underline{\boldsymbol{w}}$ will be a corresponding eigenvector, because it will satisfy

same conditions!

- $A\,(\underline{\boldsymbol{u}} - i\,\underline{\boldsymbol{w}}) = (a - b\,i)\,(\underline{\boldsymbol{u}} - i\,\underline{\boldsymbol{w}})$

$$A\vec{u} - iA\vec{w} = (a\vec{u} - b\vec{w}) + i(-b\vec{u} - a\vec{w})$$

Exercise 4 Verify that if the first eigenvector equation holds, then

$$A\,\underline{\boldsymbol{u}} = a\,\underline{\boldsymbol{u}} - b\,\underline{\boldsymbol{w}}$$
$$A\,\underline{\boldsymbol{w}} = b\,\underline{\boldsymbol{u}} + a\,\underline{\boldsymbol{w}}$$

Then check that these equalities automatically make the second conjugate eigenvector equation true as well.

Theorem Let A be a real 2×2 matrix with complex eigenvalues. Then A is similar to a rotation-dilation matrix.

proof: Let a complex eigenvalue and eigenvector be given by $\lambda = a + b\,i$, $\underline{\boldsymbol{v}} = \underline{\boldsymbol{u}} + i\,\underline{\boldsymbol{w}}$ complex, ($a, b \in \mathbb{R}$, $\underline{\boldsymbol{u}}, \underline{\boldsymbol{w}} \in \mathbb{R}^n$) Choose

$$P = [\ \mathrm{Re}\,\underline{\boldsymbol{v}} \quad \mathrm{Im}\,\underline{\boldsymbol{v}}\ ] = [\underline{\boldsymbol{u}}\ \ \underline{\boldsymbol{w}}]$$

(One can check that $\{\underline{\boldsymbol{u}}, \underline{\boldsymbol{w}}\}$ is automatically independent.) Then, using the equations of Exercise 4, we mimic what we did for diagonalizable matrices...

$$A\,[\underline{\boldsymbol{u}}\ \ \underline{\boldsymbol{w}}] = [a\,\underline{\boldsymbol{u}} - b\,\underline{\boldsymbol{w}},\ b\,\underline{\boldsymbol{u}} + a\,\underline{\boldsymbol{w}}]$$

$$= [\ \underline{\boldsymbol{u}}\ \ \underline{\boldsymbol{w}}\ ] \begin{bmatrix} a & b \\ -b & a \end{bmatrix}.$$

$$A\,P = P \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

$$P^{-1}\,A\,P = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}.$$

(The matrix on the right is a rotation-dilation matrix ... nobody ever said what the sign of b was. :-))

we'll look this over on Wednesday—

It's a mess, but one can carry out the procedure of the theorem, for the matrix B in exercise 1,

$$B = \begin{bmatrix} .9 & -.4 \\ .1 & .9 \end{bmatrix}$$

using $\lambda = .9 - .2\,i$, $\underline{v} = \begin{bmatrix} -2\,i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + i \begin{bmatrix} -2 \\ 0 \end{bmatrix}$, one gets

$$P = [\ \mathrm{Re}\,\underline{v} \quad \mathrm{Im}\,\underline{v}] = \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}, \qquad P^{-1} = \frac{1}{2}\begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}$$

$$P^{-1}BP = \frac{1}{2}\begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} .9 & -.4 \\ .1 & .9 \end{bmatrix}\begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}$$

$$P^{-1}BP = \begin{bmatrix} .9 & -.2 \\ .2 & .9 \end{bmatrix} = \sqrt{.85}\begin{bmatrix} \dfrac{.9}{\sqrt{.85}} & -\dfrac{.2}{\sqrt{.85}} \\ \dfrac{.2}{\sqrt{.85}} & \dfrac{.9}{\sqrt{.85}} \end{bmatrix}$$

$$P^{-1}BP = r\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

for $r = \sqrt{.85} \approx .92$, $\theta = \arctan\left(\frac{2}{9}\right) \approx .22$ radians.

Application on next page, to browse through...