

$$f(t) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kt) + \sum_{k=1}^{\infty} b_k \sin(kt).$$

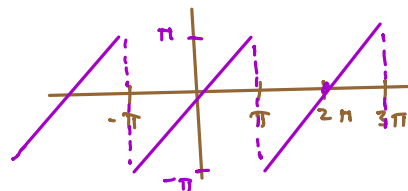
$$a_0 = \langle f, 1 \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) dt \quad a_k = \langle f, \cos(kt) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(kt) dt$$

$$b_k = \langle f, \sin(kt) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(kt) dt.$$

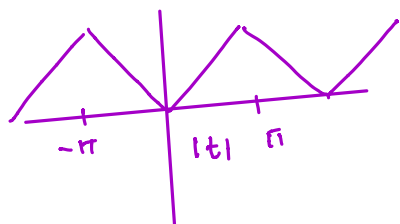
Exercise 2: Define $f(t) = t$, on the interval $-\pi < t < \pi$. Show

sawtooth

$$t \sim 2 \sum_{k=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nt)$$



in final HW
assignment
you will
study F.S.
for "fart fen"



$$f(t) = t \quad -\pi < t < \pi$$

$$a_0 = \langle f, 1 \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} t \cdot 1 dt = 0$$

$$a_n = \langle f, \cos nt \rangle$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} t \cos nt dt = 0$$

if $g(t)$ is odd
($g(-t) = -g(t)$)
e.g. poly with only
odd powers

then $\int_{-a}^a g(t) dt = 0$

$$\begin{aligned} g(t) &= t \cos nt \\ g(-t) &= (-t) \cos(-nt) \\ &= -t \cos nt \\ &= -g(t) \end{aligned}$$

g is odd.

$$b_n = \langle f, \sin nt \rangle$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} \underbrace{t}_{\text{odd}} \underbrace{\sin nt}_{\text{odd}} dt$$

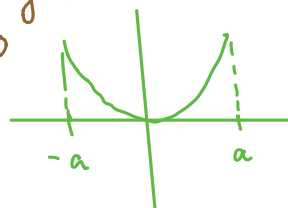
even

$$= \frac{1}{\pi} 2 \int_0^{\pi} t \sin nt$$

$\frac{2}{n} (-1)^{n+1}$

if $g(t)$ is even

$$\int_{-a}^a g(t) dt = 2 \int_0^a g(t) dt$$



$g(t)$ is even, $h(t)$ is odd
then $g(t)h(t)$ is odd.
 $g(-t)h(-t) = g(t)[-h(t)]$
 $= -g(t)h(t)$
odd \cdot odd \neq even

↓ continue on Wed.

$$b_n = \frac{2}{\pi} \int_0^\pi \underbrace{t}_u \underbrace{\sin nt \, dt}_{dv}$$

$$du = dt \quad v = -\frac{1}{n} \cos nt$$

$$\int u \, dv = uv - \int v \, du$$

$$= \frac{2}{\pi} \left[t \left(-\frac{1}{n} \cos nt \right) \right]_0^\pi + \int_0^\pi \frac{1}{n} \cos nt \, dt$$

$$= \frac{2}{\pi} \left[\cancel{\pi} \left(-\frac{1}{n} \cos n\pi \right) - 0 + \frac{1}{n^2} \sin nt \right]_0^\pi$$

$$\begin{aligned} \sin n\pi &= 0 \\ \sin 0 &= 0 \end{aligned}$$

$$= \begin{cases} 1 & n \text{ even} \\ -1 & n \text{ odd} \end{cases}$$
$$= (-1)^n$$

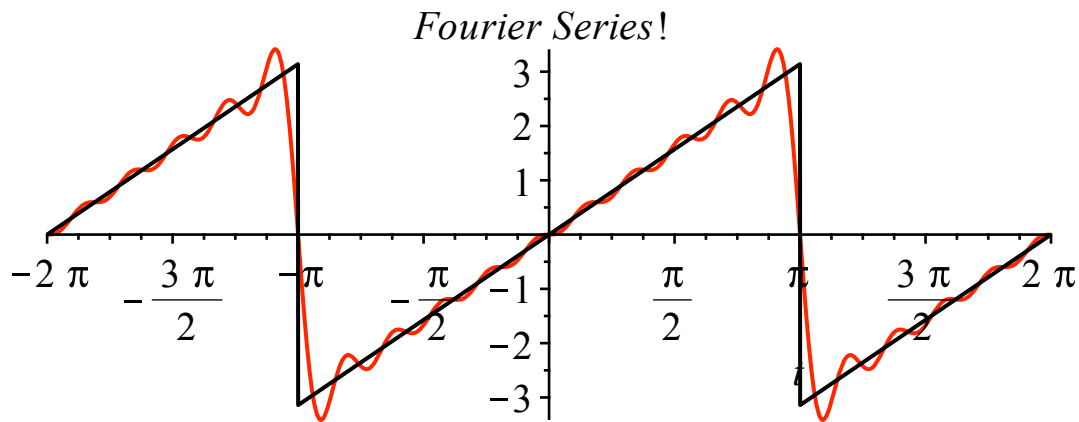
$$= \underbrace{\frac{2}{n} (-1)^{n+1}}_{b_n}$$

$$t \sim \sum_{n=1}^{\infty} \frac{2}{n} (-1)^{n+1} \sin nt$$

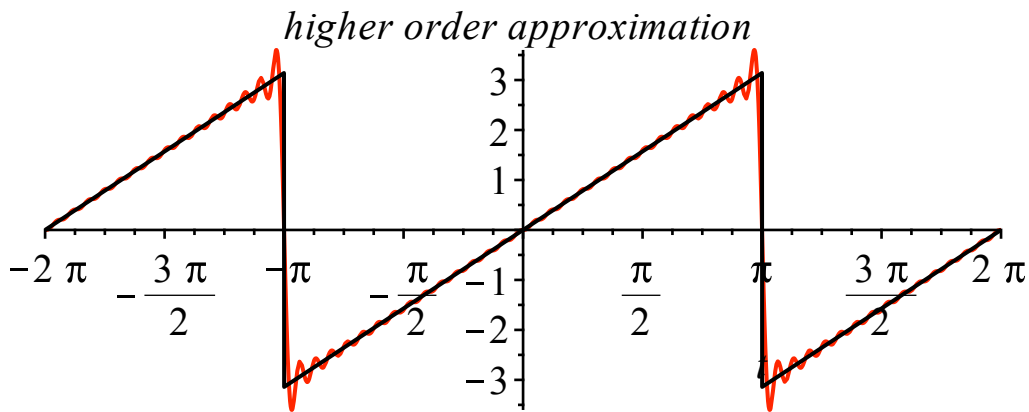
$$= 2 \left(\sin t - \frac{1}{2} \sin 2t + \frac{1}{3} \sin 3t - \dots \right)$$

$$\text{proj}_{V_{10}} f(t):$$

```
> with(plots) :
plot1 := plot(t + 2·π - 2·π·Heaviside(t + π) - 2·π·Heaviside(t - π), t = -2·π..2·π, color
= black) :
plot2 := plot(2·∑n=110 (-1)n+1 ·  $\frac{\sin(n·t)}{n}$ , t = -2·π..2·π, color = red) :
display({plot1, plot2}, title = 'Fourier Series!');
```



```
> plot3 := plot(2·∑n=130 (-1)n+1 ·  $\frac{\sin(n·t)}{n}$ , t = -2·π..2·π, color = red) :
display({plot1, plot3}, title = 'higher order approximation');
```



Wed Nov 28

- 6.8 Fourier series and applications

Announcements:

- finish F.S. example
- show & tell about .jpg image compression
- quiz : find least squares line fit

Warm-up Exercise:

Suppose $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is an orthonormal set in an inner product space. Find a formula for

$$\|a_1\vec{u}_1 + a_2\vec{u}_2 + a_3\vec{u}_3\|^2 = a_1^2 + a_2^2 + a_3^2$$

in terms of the weights a_1, a_2, a_3 .

(Recall, $\|\vec{x}\|^2 = \langle \vec{x}, \vec{x} \rangle$)

$$\begin{aligned}\|a_1\vec{u}_1 + a_2\vec{u}_2 + a_3\vec{u}_3\|^2 &= \langle a_1\vec{u}_1 + a_2\vec{u}_2 + a_3\vec{u}_3, \underbrace{a_1\vec{u}_1 + a_2\vec{u}_2 + a_3\vec{u}_3}_{\text{linearity}} \rangle \\ &= \langle a_1\vec{u}_1 + a_2\vec{u}_2 + a_3\vec{u}_3, a_1\vec{u}_1 \rangle \\ &\quad + \langle a_1\vec{u}_1 + a_2\vec{u}_2 + a_3\vec{u}_3, a_2\vec{u}_2 \rangle \\ &\quad + \langle a_1\vec{u}_1 + a_2\vec{u}_2 + a_3\vec{u}_3, a_3\vec{u}_3 \rangle \\ &= a_1^2 \langle \vec{u}_1, \vec{u}_1 \rangle + \cancel{a_2 a_1 \langle \vec{u}_2, \vec{u}_1 \rangle} + \cancel{a_3 a_1 \langle \vec{u}_3, \vec{u}_1 \rangle} \\ &\quad + a_2^2 \cancel{\langle \vec{u}_2, \vec{u}_2 \rangle} + \cancel{a_3 a_2 \langle \vec{u}_3, \vec{u}_2 \rangle} \\ &\quad + a_3^2 \langle \vec{u}_3, \vec{u}_3 \rangle \\ &= a_1^2 + a_2^2 + a_3^2\end{aligned}$$

As part of the deep theorem about Fourier series, as long as f is piecewise continuous,

$$\left\| \text{proj}_{V_N} f - f \right\| \rightarrow 0 \quad \text{and} \quad \left\| \text{proj}_{V_N} f \right\| \rightarrow \|f\|.$$

Recall, the norm that we get from the Fourier series inner product is

$$\|g\|^2 = \frac{1}{\pi} \int_{-\pi}^{\pi} g(t)^2 dt.$$

Now,

$$\text{proj}_{V_n} f = \frac{a_0}{2} + \sum_{k=1}^n a_k \cos(kt) + \sum_{k=1}^n b_k \sin(kt)$$

So

$$\begin{aligned} \left\| \text{proj}_{V_n} f \right\|^2 &= \left\| \frac{a_0}{2} + \sum_{k=1}^n a_k \cos(kt) + \sum_{k=1}^n b_k \sin(kt) \right\|^2 \\ &= \left\langle \frac{a_0}{2} + \sum_{k=1}^n a_k \cos(kt) + \sum_{k=1}^n b_k \sin(kt), \frac{a_0}{2} + \sum_{k=1}^n a_k \cos(kt) + \sum_{k=1}^n b_k \sin(kt) \right\rangle \\ &= \left\langle \frac{a_0}{2}, \frac{a_0}{2} \right\rangle + \sum_{k=1}^n a_k^2 + \sum_{k=1}^n b_k^2 = \frac{a_0^2}{2} + \sum_{k=1}^n a_k^2 + \sum_{k=1}^n b_k^2, \end{aligned}$$

because the cross terms in the expanded inner product cancel out - since the basis vectors we've chosen for V_n are orthonormal:

$$V_n := \text{span} \left\{ \frac{1}{\sqrt{2}}, \cos(t), \sin(t), \cos(2t), \sin(2t), \dots, \cos(nt), \sin(nt) \right\}$$

As an application, for our function $f(t) = t$,

$$\|f\|^2 = \frac{1}{\pi} \int_{-\pi}^{\pi} t^2 dt = \frac{2}{\pi} \int_0^{\pi} t^2 dt = \frac{2}{3} \pi^2.$$

Since

$$t \sim t \sim 2 \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \sin(kt)$$

It must be that

$$\frac{2}{3} \pi^2 = 4 \sum_{k=1}^{\infty} \frac{1}{k^2}$$

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots \quad \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

This magic formula is true (which is sort of amazing), although you may not have seen it before:

$$\left[> \sum_{k=1}^{\infty} \frac{1}{k^2}; \right.$$

$$\frac{1}{6} \pi^2$$

(1)

Image compression show and tell... I'll demo the matlab code

Math 3150 2D Fourier Series Field Trip Project

Due date: Monday, Nov. 28 after Thanksgiving holiday.

Two-dimensional Fourier series can be used to perform image processing and data compression, and is a salient example of how an set of orthogonal basis functions can be used to approximate functions. Suppose $f(x, y)$ is defined on the region $(x, y) \in [0, L] \times [0, H]$ and represents a grey scale image. For each point (x, y) the greyscale value ranges from zero (black) to unity (white) $f \in [0, 1]$. The orthogonal basis set we use is a 2D sine series:

$$\phi_{n,m}(x, y) = \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{m\pi}{H}y\right),$$

.jpg uses cosine series instead.

where n and m both range from $1, 2, 3, \dots$. The values $\frac{n\pi}{L}$ and $\frac{m\pi}{H}$ represent the horizontal and vertical spatial frequencies. The approximate image is the double sum orthogonal projection:

$$\hat{f}_{N,M}(x, y) = \sum_{n=1}^N \sum_{m=1}^M B_{n,m} \phi_{n,m}(x, y),$$

where N and M represent the sum truncation and the values $\frac{N\pi}{L}$ and $\frac{M\pi}{H}$ represent the maximum horizontal and vertical spatial frequencies available to represent the image—any image feature that has higher spatial frequency, such as sharp areas of contrast, fine texture, etc, cannot be represented. The Fourier coefficients are

$$B_{n,m} = \frac{\langle f, \phi_{n,m} \rangle}{\langle \phi_{n,m}, \phi_{n,m} \rangle}, \quad \langle g, h \rangle = \int_0^L \int_0^H g(x, y) h(x, y) dy dx.$$

The goal of the project is to assess the qualitative nature of Fourier image processing in two experiments.

Experiment 1:

The first experiment you will show a subject (a friend that has not seen the full image) a Fourier-decomposed image of a car/truck that you take with your cell phone camera. I advise to use a "square" Instagram-ready image, and compress it to 256X256 pixels, which is most easily accomplished by re-sending the picture to yourself by email and compressing it to the "small" size for sending. The automobile image should be centered in the frame, and fill approximately 1/2-3/4 the width of the frame. It should be a random car parked on a street, or something, that's from a common brand and model that's recognizable to most people, or at least your friend. Your friend should not know anything about the picture at all and don't tell them anything. Show your friend successively higher truncated Fourier compressed images of the car until your friend correctly guesses (1) that its a automobile of some type, and then (2) guesses the make and/or model type. Start by showing your friend the Fourier compressed image at $N = M = 10$, then ascend $N = 20, 30, 40, 50, \dots$ until he/she gets both (1) and then (2). At each stage, record you friend's response and report your results, including the images and the compressed images.

Experiment 2:

Take two pictures, both 256X256 as described above. One of the images should be a "natural scene," which should be interpreted broadly as naturescapes, or varied urban cityscapes—the point is that it should contain a mix lots of things in the image, both foreground and background, objects with lots of different sizes in the frame—and the other should be a somewhat boring picture of a single human-made material—e.g., a wall of bricks, patterned fabric, things of a regular or repeated nature to it; be sure that you fit several repetitions of the pattern in your picture.. Get inventive with what you choose. We will compare the two pictures' Fourier coefficients $B_{n,m}$. Natural images have been commonly reported to have squared Fourier coefficients that decay with a power law:

$$B_{n,m}^2 \sim \frac{1}{n^\gamma} \quad \text{or} \quad \sim \frac{1}{m^\gamma},$$

where γ is typically in the range between 1.7 and 2.3. That is, γ is usually around 2. Why do we examine squared Fourier coefficients? Its because squared values are associated with the energy in the image through Parseval's identity:

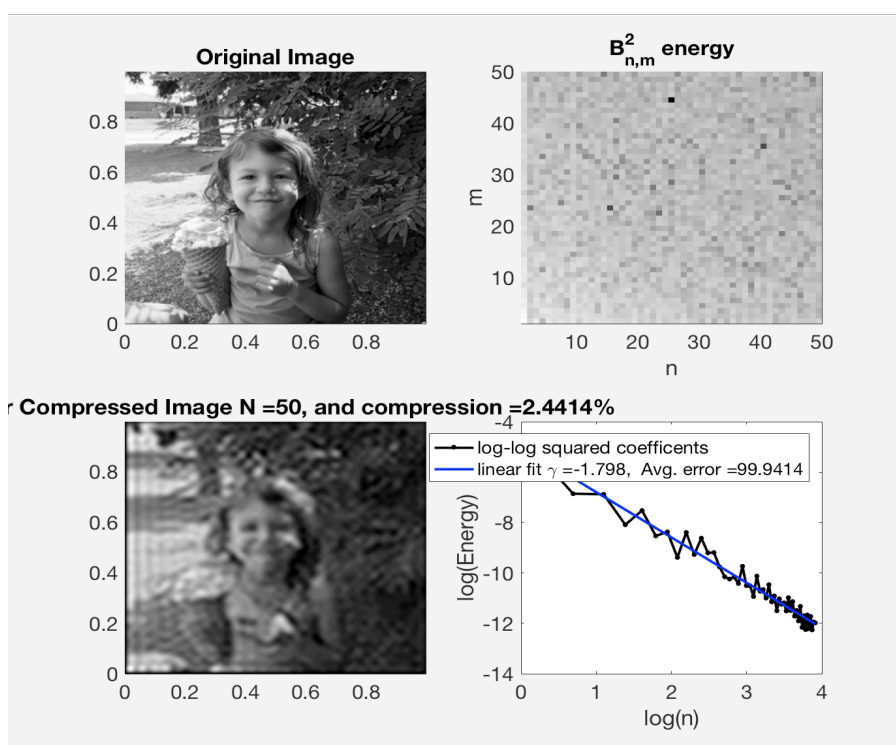
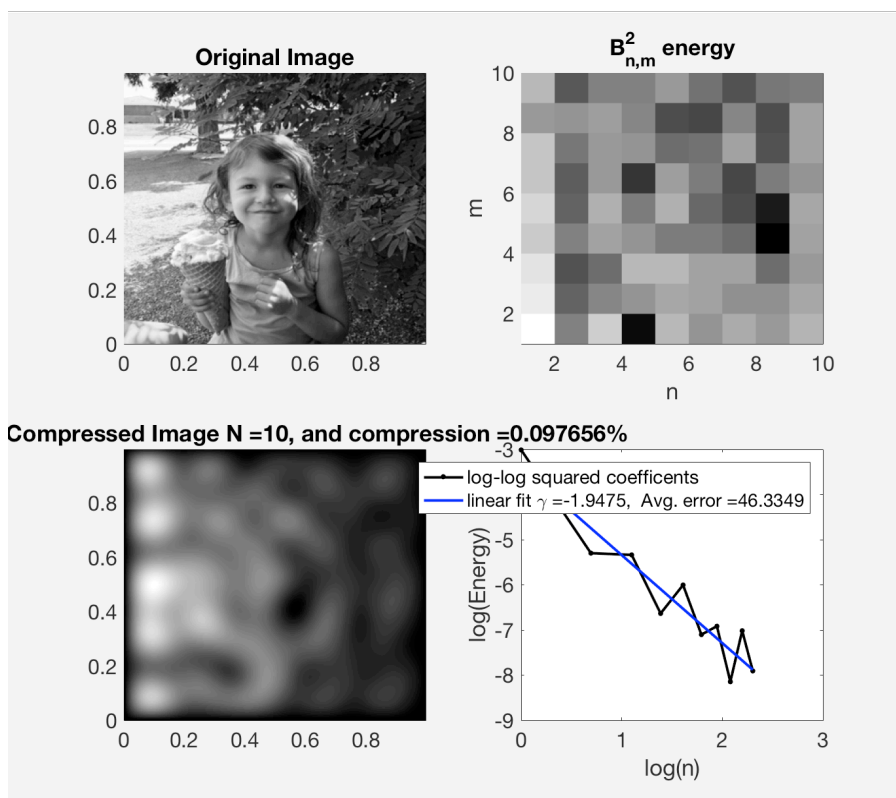
$$Energy = \int_0^L \int_0^H |f(x, y)|^2 dy dx = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} B_{n,m}^2.$$

In the accompanying code, the coefficients $B_{n,m}$ are computed and represented as a matrix, and rendered of the squared values of the $B_{n,m}$. The energy spectrum is a log-log plot of the average of vertical and horizontal average energies: $\frac{1}{2}avg_m(B_{n,m}^2) + \frac{1}{2}avg_m(B_{m,n}^2) = b_n^2$ —this gives an estimate of the spectral energy at each spatial frequency $n\pi$ per unit image length L . If $b_n^2 \sim \frac{1}{n^\gamma}$, then taking the log of both sides we get:

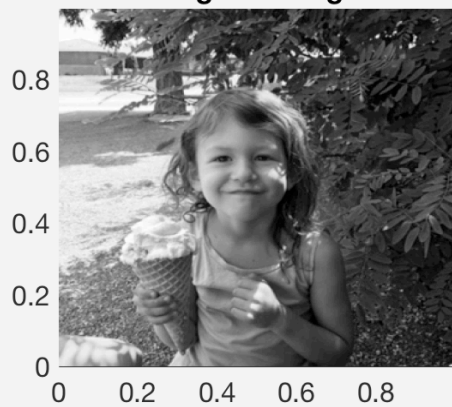
$$\ln(b_n^2) \sim \ln\left(\frac{1}{n^\gamma}\right) = -\gamma \ln(n).$$

That is, the log squared coefficient averages will be linearly related to the log of n with slope $-\gamma$. The code performs a linear curve fitting on the log-log data and finds the best-fit γ -value as our estimate. For the two images you choose, record the γ -estimates and report them in your results along with your images. Use a large truncation $N = 100$ value—it may take a while. Report the gamma-values you find, and the standard deviations of the linear fit.

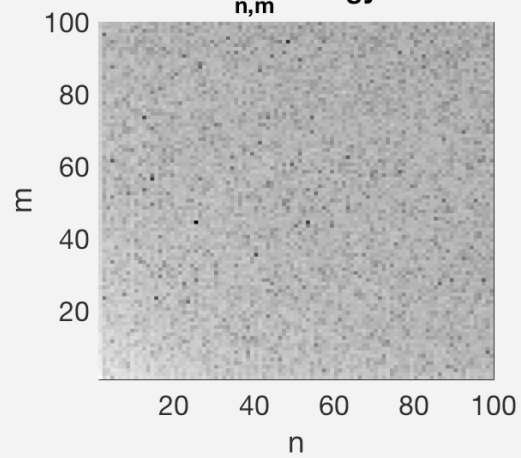
How to use the code: Put the images you want to analyze in a file folder with the .m code given with this experiment. Type in the code the file name of the image you want to examine and edit the code to set an N value for your truncation. There is a variable called "experiment", which you set to 1 or 2, respectively. The code will output figures. Figure 1 will give you the N th Fourier truncated image in greyscale. Figure 2 will output the results for experiment 2.



Original Image



$B_{n,m}^2$ energy



Compressed Image N =100, and compression =9.7656%

