Definition: Let $A_{n\times n}$. If there is an $\mathbb{R}^n$ (or $\mathbb{C}^n$) basis $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots, \underline{\boldsymbol{v}}_n$ consisting of eigenvectors of A, then A is called diagonalizable. This is precisely why:

Write $A\,\underline{\boldsymbol{v}}_j = \lambda_j\,\underline{\boldsymbol{v}}_j$ (some of these λ_j may be the same, as in the previous example). Let P be the matrix

$$P = \left[\underline{\boldsymbol{v}}_1 \middle| \underline{\boldsymbol{v}}_2 \middle| \cdots \middle| \underline{\boldsymbol{v}}_n\right].$$

Then, using the various ways of understanding matrix multiplication, we see

$$A\,P = A\left[\underline{\boldsymbol{v}}_1 \middle| \underline{\boldsymbol{v}}_2 \middle| \cdots \middle| \underline{\boldsymbol{v}}_n\right] = \left[\lambda_1\underline{\boldsymbol{v}}_1 \middle| \lambda_2\underline{\boldsymbol{v}}_2 \middle| \cdots \middle| \lambda_n\underline{\boldsymbol{v}}_n\right]$$

$$= \left[\underline{\boldsymbol{v}}_1 \middle| \underline{\boldsymbol{v}}_2 \middle| \cdots \middle| \underline{\boldsymbol{v}}_n\right] \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ : & : & \ldots & : \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix}.$$

$$A\,P = P\,\mathrm{D}$$
$$A = P\,\mathrm{D}\,P^{-1}$$
$$P^{-1}A\,P = \mathrm{D}\,.$$

Unfortunately, as we've already seen, not all matrices are diagonalizable:

Exercise 2) Show that

$$C := \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

is not diagonalizable. (Even though it has the same characteristic polynomial as B, which was diagonalizable.

Fri Nov 2

- 5.4 eigenvalues, eigenvectors and linear transformations

Announcements:

- posted HW; exam next Friday includes these topics (since last midterm material)
- Pac12 extra credit on HW (10 pts).
- review Wed "diagonalization" then today's notes

review!

Warm-up Exercise: Let $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$ be a basis for $\mathbb{R}^2$

(a) If $[\vec{x}]_\beta = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ find $\vec{x}$. $(= [\vec{x}]_E)$

$$\vec{x} = 1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$$

in general,

$$\vec{x} = \underbrace{\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}}_{\text{"}P\text{"}} [\vec{x}]_\beta$$

"$P_{E \leftarrow \beta}$"

(b) If $\vec{x} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$ find $[\vec{x}]_\beta$

$$c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 2 & 1 & 3 \\ 1 & -1 & 0 \end{array}\right]$$

$R_1 \leftrightarrow R_2$
$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 2 & 1 & 3 \end{array}\right]$$

$-2R_1 + R_2 \to R_2$
$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 3 & 3 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right]$$

$$[\vec{x}]_\beta = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

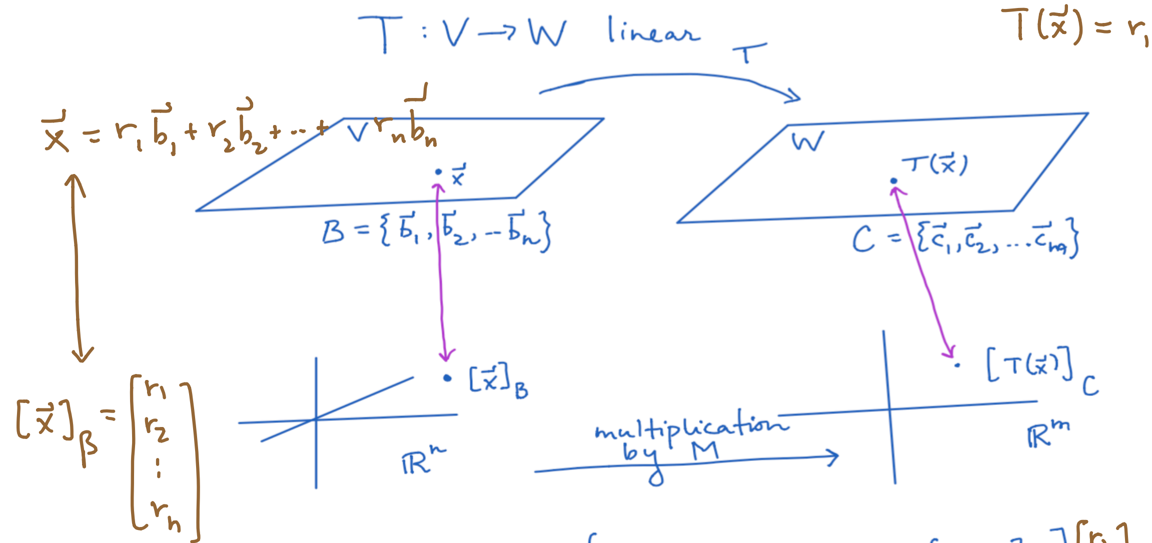
$$1 \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \checkmark$$

OR use $\vec{x} = P[\vec{x}]_\beta$

$$\underbrace{P^{-1}}_{P_{\beta \leftarrow E}} \vec{x} = [\vec{x}]_\beta$$

you'd get same answer.

If we have a linear transformation $T : V \rightarrow W$ and bases $B = \{\underline{b}_1, \underline{b}_2, \ldots \underline{b}_n\}$ in V, $C = \{\underline{c}_1, \underline{c}_2, \ldots \underline{c}_m\}$ in W, then the matrix of T with respect to these two bases transforms the B coordinates of vectors $\underline{v} \in V$ to the C coordinates of $T(\underline{v})$ in a straightforward way, although it takes a while to get used to:



since T is linear

$T(\vec{x}) = r_1 T(\vec{b}_1) + r_2 T(\vec{b}_2) + \cdots + r_n T(\vec{b}_n)$

$[T(\vec{x})]_C$

$= r_1 [T(\vec{b}_1)]_C + r_2 [T(\vec{b}_2)]_C + \cdots + r_n [T(\vec{b}_n)]_C$

$//$

$$M = \begin{bmatrix} [T(\vec{b}_1)]_C & [T(\vec{b}_2)]_C & \cdots & [T(\vec{b}_n)]_C \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix}$$

<u>Exercise 1</u>) Let $V = P_3 = span\{1, t, t^2, t^3\}$, $W = P_2 = span\{1, t, t^2\}$, and let $D : V \rightarrow W$ be the derivative operator. Find the matrix of D with respect to the bases $\underbrace{\{1, t, t^2, t^3\}}_{B}$ in V and $\underbrace{\{1, t, t^2\}}_{C.}$ in W. Test your result.

use recipe: $M = \begin{bmatrix} [D_t(1)]_C & [D_t(t)]_C & [D(t^2)]_C & [D(t^3)]_C \end{bmatrix}$

since $D_t 1 = 0 \quad D_t t = 1 \quad D_t t^2 = 2t \quad D t^3 = 3t^2$

take words to get cols.

$$M = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

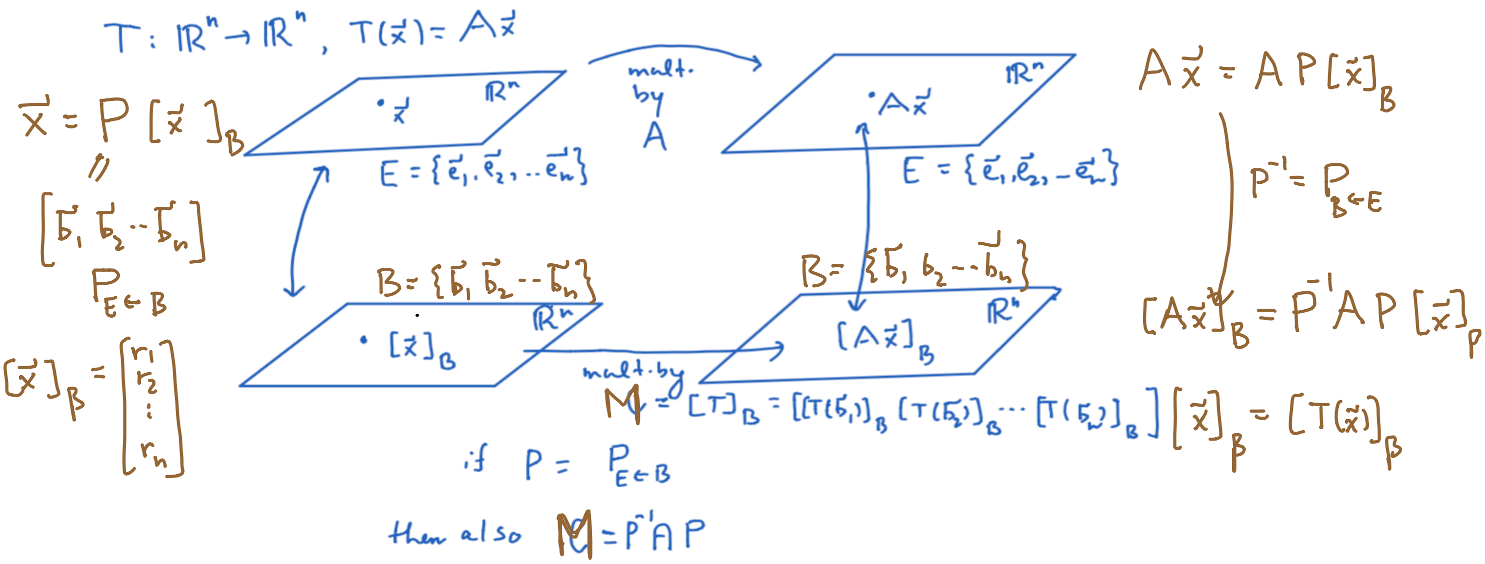
now, use the matrix to find $D(4 + 2t - 7t^2 + 3t^3)$

Ans. $[D\,p(t)]_C = M[p(t)]_B$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ -7 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -14 \\ 9 \end{bmatrix}$$

<u>So</u> $D(p(t)) = 2 - 14t + 9t^2$

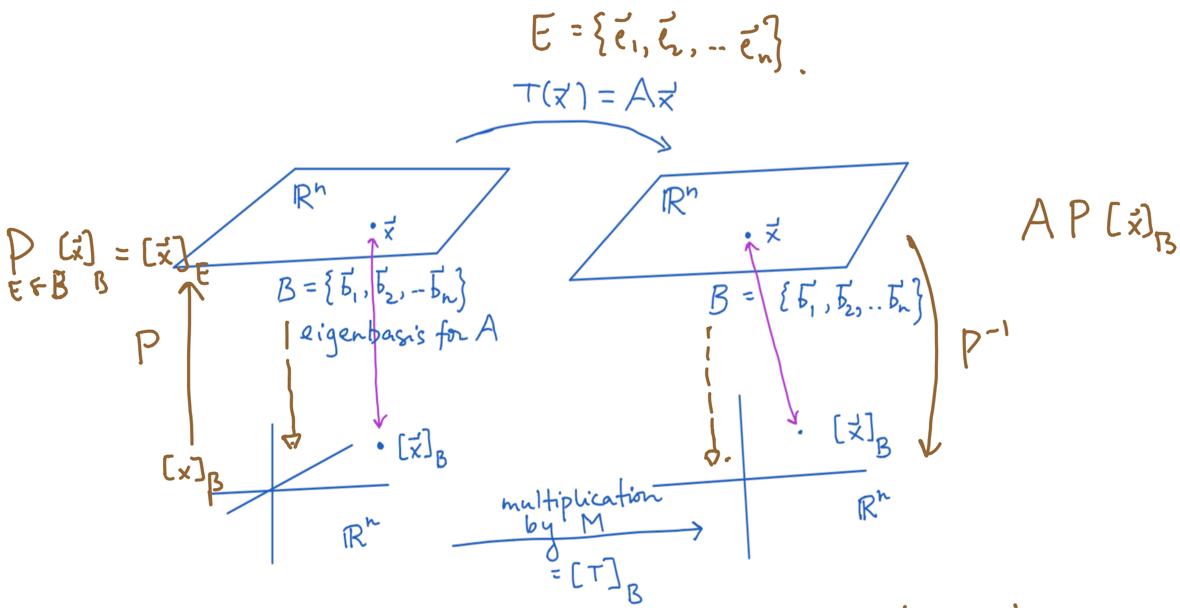
A special case of the previous page is when $T : \mathbb{R}^n \to \mathbb{R}^n$ is a matrix transformation $T(\underline{\boldsymbol{x}}) = A\,\underline{\boldsymbol{x}}$, and we find the matrix of T with respect to a non-standard basis (the same non-standard basis in the domain and in the codomain).



<u>Definition</u> Two matrices A, B are called *similar* if there is an invertible matrix P with $B = P^{-1}AP$. As the diagram above shows, similar matrices arise when one is describing the same linear transformation, but with respect to different bases.

algebraically means $P^{-1}AP = D$ diagonal, where P is a matrix of eigenvectors of A (& evects are a basis for $\mathbb{R}^n$).

Exercise 2) What if a matrix A is diagonalizable? What is the matrix of $T(\underline{\boldsymbol{x}}) = A\,\underline{\boldsymbol{x}}$ with respect to the eigenbasis? How does this connect to our matrix identities for diagonalization? Fill in the matrix M below, and then compute another way to express it, as a triple product using the diagram.



$B = \{\vec{b}_1, \vec{b}_2 \cdots \vec{b}_n\} \qquad A\vec{b}_j = \lambda_j \vec{b}_j$

$$M = [T]_B = \left[[T(\vec{b}_1)]_B \;\; [T(\vec{b}_2)]_B \cdots [T(\vec{b}_n)]_B \right]$$

since $T(\vec{b}_1) = \lambda_1 \vec{b}_1, \; T(\vec{b}_2) = \lambda_2 \vec{b}_2$

$$M = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & \vdots \\ \vdots & \vdots & \\ 0 & 0 & \lambda_n \end{bmatrix} = \text{"}D\text{"} = P^{-1}AP$$

Example, from earlier this week:

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} \qquad E_{\lambda=4} = span\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\} \qquad E_{\lambda=1} = span\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\} \qquad \boxed{B = \left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}}$$

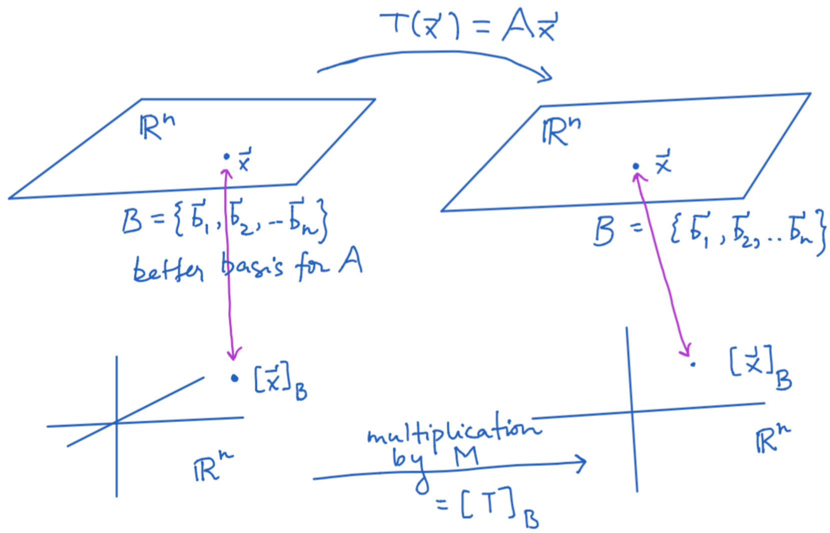
Write the various matrices corresponding to the diagram above.

$$P_{E \leftarrow B} = P = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}.$$

$$M = D = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} = M = [T]_B = P^{-1}AP$$

Even if the matrix A is not diagonalizable, there may be a better basis to help understand the transformation $T(\underline{\mathbf{x}}) = A\,\underline{\mathbf{x}}$. The diagram on the previous page didn't require that B be a basis of eigenvectors....maybe it was just a "better" basis than the standard basis, to understand T.



<u>Exercise 3</u> (If we have time - this one is not essential.) Try to pick a better basis to understand the matrix transformation $T(\underline{\mathbf{x}}) = C\,\underline{\mathbf{x}}$, even though the matrix C is not diagonalizable. Compute $M = P^{-1} A\, P$ or compute M directly, to see if it really is a "better" matrix.

$$C = \begin{bmatrix} 4 & 4 \\ -1 & 0 \end{bmatrix}$$