Math 2270-002 Week 13 notes
We will not necessarily finish the material from a given day's notes on that day. We may also add or subtract some material as the week progresses, but these notes represent an in-depth outline of what we plan to cover. These notes cover material in 6.4-6.6

Mon Nov 19 finish

- 6.4 Gram Schmidt and $A = QR$ decomposition. Orthogonal matrices Q

Announcements:
- Wed is a real class day
- there is a quiz (but I'll up the # of dropped quizzes to 3)
- HW is due (I'll accept emailed copies)

Warm-up Exercise: Compute $\text{proj}_W \begin{bmatrix} 5 \\ -2 \\ 7 \\ -6 \end{bmatrix}$, $W = \text{span}\{\vec{e}_1, \vec{e}_2, \vec{e}_4\}$

$\begin{bmatrix} 5 \\ -2 \\ 0 \\ -6 \end{bmatrix}$ is what you expect

$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

using $\text{proj}_W \vec{x} = (\vec{x}\cdot\vec{u}_1)\vec{u}_1 + (\vec{x}\cdot\vec{u}_2)\vec{u}_2 + \ldots + (\vec{x}\cdot\vec{u}_p)\vec{u}_p$

for $\{\vec{u}_1, \vec{u}_2, \ldots \vec{u}_p\}$ orthonormal basis of W

$$\text{proj}_W \vec{x} = (\vec{x}\cdot\vec{e}_1)\vec{e}_1 + (\vec{x}\cdot\vec{e}_2)\vec{e}_2 + (\vec{x}\cdot\vec{e}_4)\vec{e}_4$$
$$= 5\vec{e}_1 + (-2)\vec{e}_2 - 6\vec{e}_4$$
$$= \begin{bmatrix} 5 \\ -2 \\ 0 \\ -6 \end{bmatrix} \checkmark$$

We begin on Monday with a continuation of the discussion of Gram-Schmidt orthogonalization from 6.4. Keeping track of the G.S. process carefully yields the $A = QR$ matrix product decomposition theorem, where Q is an "orthogonal matrix" consisting of an orthonormal basis for the span of the columns of A and R is an upper triangular matrix with positive entries along the diagonal. This decomposition is one way to understand why matrix determinants correspond to $\pm$ Volumes, in $\mathbb{R}^n$, and can also be useful in solving multiple linear systems of equations with the same "A" matrix more efficiently.

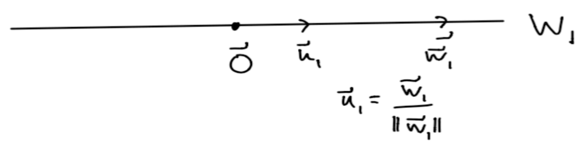
Section 6.5, *Least square solutions* is about finding approximate solutions to inconsistent matrix equations, and relies on many of the ideas we've been studying in Chapter 6 up to this point. Whenever one tries to fit experimental data to finite dimensional models it is extremely unlikely that one will get an exact fit. Least squares solutions are the "best possible", and for this reason software like Matlab automatically returns the least squares "solution" when aske to solve an inconsistent system.

Section 6.6, *Applications to linear models,* is an application of the least squares method to e.g. single or multivariate linear regression in statistics.

Recall the Gram-Schmidt process from Friday:

Start with a basis $B = \{\underline{w}_1, \underline{w}_2, \ldots \underline{w}_p\}$ for a subspace W of $\mathbb{R}^n$. How can you convert it into an orthonormal basis? Here's how! The inductive process is called Gram-Schmidt orthogonalization.

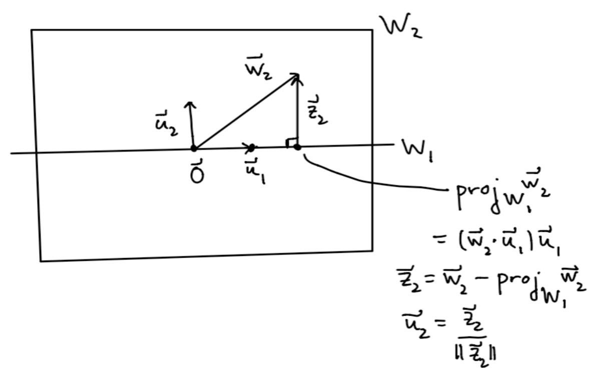
Let $W_1 = span\{\underline{w}_1\}$. Define $\underline{u}_1 = \dfrac{\underline{w}_1}{\|\underline{w}_1\|}$. Then $\{\underline{u}_1\}$ is an orthonormal basis for W_1.



Let $W_2 = span\{\underline{w}_1, \underline{w}_2\} = span\{\underline{u}_1, \underline{w}_2\}$.

Let $\underline{z}_2 = \underline{w}_2 - proj_{W_1}\underline{w}_2 = \underline{w}_2 - (\underline{w}_2 \cdot \underline{u}_1)\underline{u}_1$ so $\underline{z}_2 \perp \underline{u}_1$.

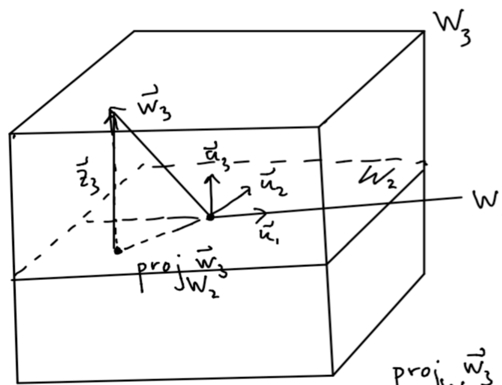
Define $\underline{u}_2 = \dfrac{\underline{z}_2}{\|\underline{z}_2\|}$. So $\{\underline{u}_1, \underline{u}_2\}$ is an orthonormal basis for W_2.



Let $W_3 = span\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$.

Let $\mathbf{z}_3 = \mathbf{w}_3 - proj_{W_2}\mathbf{w}_3$, so $\mathbf{z}_3 \perp W_2$.

Define $\mathbf{u}_3 = \dfrac{\mathbf{z}_3}{\|\mathbf{z}_3\|}$. Then $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal basis for W_3.



$proj_{W_2}\vec{w}_3 = (\vec{w}_3\cdot\vec{u}_1)\vec{u}_1 + (\vec{w}_3\cdot\vec{u}_2)\vec{u}_2$

$\vec{z}_3 = \vec{w}_3 - proj_{W_2}\vec{w}_3$

$\vec{u}_3 = \dfrac{\vec{z}_3}{\|\vec{z}_3\|}$

Inductively,

Let $W_j = span\{\mathbf{w}_1, \mathbf{w}_2, \ldots \mathbf{w}_j\} = span\{\mathbf{u}_1, \mathbf{u}_2, \ldots \mathbf{u}_{j-1}, \mathbf{w}_j\}$.

Let $\mathbf{z}_j = \mathbf{w}_j - proj_{W_{j-1}}\mathbf{w}_j = \mathbf{w}_j - (\mathbf{w}_j \cdot \mathbf{u}_1)\mathbf{u}_1 - (\mathbf{w}_j \cdot \mathbf{u}_2)\mathbf{u}_2 - \ldots (\mathbf{w}_j \cdot \mathbf{u}_{j-1})\mathbf{u}_{j-1}$.

Define $\mathbf{u}_j = \dfrac{\mathbf{z}_j}{\|\mathbf{z}_j\|}$. Then $\{\mathbf{u}_1, \mathbf{u}_2, \ldots \mathbf{u}_j\}$ is an orthonormal basis for W_j .

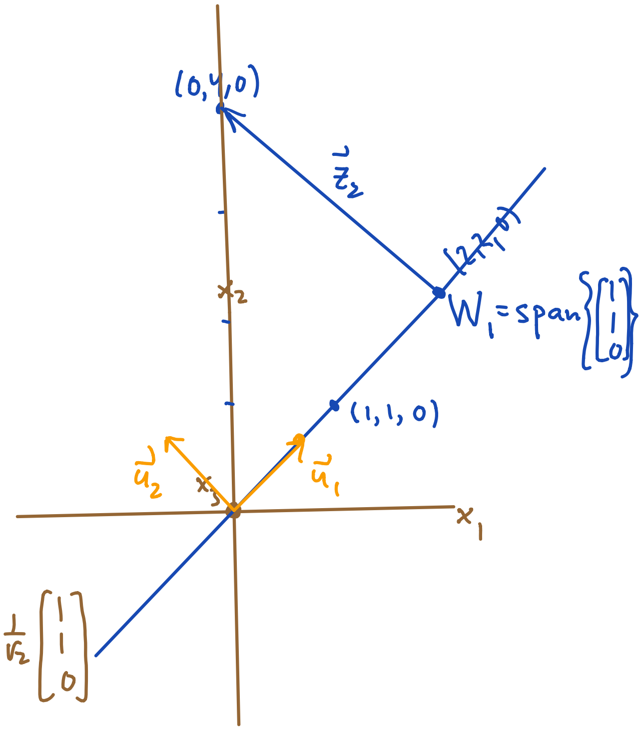
Continue up to $j = p$.

Exercise 1 Perform Gram-Schmidt on the $\mathbb{R}^3$ basis

$$B = \left\{ \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{\vec{w}_1}, \underbrace{\begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix}}_{\vec{w}_2}, \underbrace{\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}}_{\vec{w}_3} \right\}.$$

This will proceed as the Friday exercise until the third step, i.e.

$$\boldsymbol{u}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{u}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$



$$\vec{u}_1 = \frac{\vec{w}_1}{\|\vec{w}_1\|} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\vec{z}_2 = \vec{w}_2 - \text{proj}_{W_1}\vec{w}_2$$

$$= \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix} - (\vec{w}_2 \cdot \vec{u}_1)\vec{u}_1 = \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix} - \left(\begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right) \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix}$$

$$\vec{u}_2 = \frac{\vec{z}_2}{\|\vec{z}_2\|} = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$\vec{z}_3 = \vec{w}_3 - \text{proj}_{W_2}\vec{w}_3$$

$$\vec{z}_3 = \vec{w}_3 - (\vec{w}_3 \cdot \vec{u}_1)\vec{u}_1 - (\vec{w}_3 \cdot \vec{u}_2)\vec{u}_2 \longleftarrow$$

how to remember: want $\vec{z}_3 \cdot \vec{u}_1 = 0$
$\vec{z}_3 \cdot \vec{u}_2 = 0$

$$= \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} - \underbrace{\left(\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right) \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{-\frac{1}{2}(-1)\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}} - \underbrace{\left(\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right) \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}_{-\frac{1}{2}(-3)\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}$$

$$= \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{3}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$$

$$\vec{u}_3 = \frac{\vec{z}_3}{\|\vec{z}_3\|} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\{\vec{u}_1, \vec{u}_2, \vec{u}_3\} = \left\{ \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

recall Fri $\{\vec{u}_1, \vec{u}_2, \ldots \vec{u}_k\}$ o.n. basis for V

if $\vec{x} \in V$, $\vec{x} = (x \cdot \vec{u}_1)\vec{u}_1 + \ldots (\vec{x} \cdot \vec{u}_k)\vec{u}_k$

$A = Q\,R$ decomposition:

We're denoting the original basis for W by $B = \{\underline{\boldsymbol{w}}_1, \underline{\boldsymbol{w}}_2, \ldots \underline{\boldsymbol{w}}_p\}$. Denote the orthonormal basis we've constructed with Gram-Schmidt by $O = \{\underline{\boldsymbol{u}}_1, \underline{\boldsymbol{u}}_2, \ldots \underline{\boldsymbol{u}}_p\}$. Because O is orthonormal it's easy to express these two bases in terms of each other. Notice

$$W_j = span\ \{\underline{\boldsymbol{w}}_1, \underline{\boldsymbol{w}}_2, \ldots \underline{\boldsymbol{w}}_j\} = span\ \{\underline{\boldsymbol{u}}_1, \underline{\boldsymbol{u}}_2, \ldots \underline{\boldsymbol{u}}_j\} \qquad \text{for each } 1 \le j \le p.$$

So,

$$\underline{\boldsymbol{w}}_1 = (\underline{\boldsymbol{w}}_1 \bullet \underline{\boldsymbol{u}}_1)\underline{\boldsymbol{u}}_1$$

$$\underline{\boldsymbol{w}}_2 = (\underline{\boldsymbol{w}}_2 \bullet \underline{\boldsymbol{u}}_1)\underline{\boldsymbol{u}}_1 + (\underline{\boldsymbol{w}}_2 \bullet \underline{\boldsymbol{u}}_2)\underline{\boldsymbol{u}}_2$$

$$\vdots$$

$$\underline{\boldsymbol{w}}_j = (\underline{\boldsymbol{w}}_j \bullet \underline{\boldsymbol{u}}_1)\underline{\boldsymbol{u}}_1 + (\underline{\boldsymbol{w}}_j \bullet \underline{\boldsymbol{u}}_2)\underline{\boldsymbol{u}}_2 + \ldots\ (\underline{\boldsymbol{w}}_j \bullet \underline{\boldsymbol{u}}_j)\underline{\boldsymbol{u}}_j$$

$$\vdots$$

$$\underline{\boldsymbol{w}}_p = \sum_{l=1}^{p} (\underline{\boldsymbol{w}}_l \bullet \underline{\boldsymbol{u}}_l)\underline{\boldsymbol{u}}_l\ .$$

Notice that the coefficients of the last terms in the sums above, namely $(\underline{\boldsymbol{w}}_j \bullet \underline{\boldsymbol{u}}_j)$ can be computed as

$$(\underline{\boldsymbol{w}}_j \bullet \underline{\boldsymbol{u}}_j) = \underline{\boldsymbol{z}}_j \bullet \frac{\underline{\boldsymbol{z}}_j}{\|\underline{\boldsymbol{z}}_j\|} = \|\underline{\boldsymbol{z}}_j\|.$$

In matrix form (column by column) we have

$$* \quad \underbrace{\begin{bmatrix} \vdots & \vdots & & \vdots \\ \vec{w}_1 & \vec{w}_2 & \cdots & \vec{w}_p \\ \vdots & \vdots & & \vdots \end{bmatrix}}_{\substack{\text{"A"} \\ \text{columns are} \\ \text{original basis} \\ \text{for } W \\ A_{n \times p}}} = \underbrace{\begin{bmatrix} \vdots & \vdots & & \vdots \\ \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_p \\ \vdots & \vdots & & \vdots \end{bmatrix}}_{\substack{\text{"Q"} \\ \text{columns are} \\ \text{orthonormal} \\ Q_{n \times p}}} \underbrace{\begin{bmatrix} \vec{w}_1 \cdot \vec{u}_1 & \vec{w}_2 \cdot \vec{u}_1 & \vec{w}_3 \cdot \vec{u}_1 & \cdots & \vec{w}_p \cdot \vec{u}_1 \\ 0 & \vec{w}_2 \cdot \vec{u}_2 & \vec{w}_3 \cdot \vec{u}_2 & & \vec{w}_p \cdot \vec{u}_2 \\ 0 & 0 & \vec{w}_3 \cdot \vec{u}_3 & \ddots & \vdots \\ \vdots & \vdots & 0 & & \\ 0 & 0 & 0 & & \vec{w}_p \cdot \vec{u}_p \end{bmatrix}}_{\substack{\text{"R"} \\ \text{upper } \Delta\text{'ular, with} \\ \text{diagonal entries} \\ \vec{w}_j \cdot \vec{u}_j = \|\vec{z}_j\| \\ R_{p \times p}}}$$

Thus any matrix with linearly independent columns may be written in factored form as above, ($W = Col\,A$),

$$A_{n \times p} = Q_{n \times p} R_{p \times p}.$$

This factorization contains geometric information and can simplify the computational work needed to solve matrix equations $A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}$.

From previous page...

$$* \qquad A_{n \times p} = Q_{n \times p} R_{p \times p}$$

<u>shortcut</u> (or what to do if you forgot the formulas for the entries of R) If you just know Q you can recover R by multiplying both sides of the $*$ equation on the previous page by the transpose Q^T of the Q matrix:

$$\underset{Q^T}{\begin{bmatrix} \text{---} \ \vec{u}_1^T \ \text{---} \\ \text{---} \ \vec{u}_2^T \ \text{---} \\ \vdots \\ \text{---} \ \vec{u}_p^T \ \text{---} \end{bmatrix}} \underset{A}{\begin{bmatrix} \vec{w}_1 & \vec{w}_2 & \cdots & \vec{w}_p \end{bmatrix}} = \underset{\cancel{Q^T}}{\begin{bmatrix} \text{---} \ \vec{u}_1^T \ \text{---} \\ \text{---} \ \vec{u}_2^T \ \text{---} \\ \vdots \\ \vec{u}_p^T \end{bmatrix}} \underset{Q}{\begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_p \end{bmatrix}} \underset{R}{R} = IR = R!$$

$Q^T Q = I$ because $\{\vec{u}_1, \vec{u}_2, \ldots \vec{u}_p\}$ are orthonormal

$$A = Q\,R$$
$$Q^T A = Q^T Q\,R = I\,R = R.$$

<u>Example)</u> From last Friday,

$$B = \left\{ \underset{\vec{w}_1}{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}, \underset{\vec{w}_2}{\begin{bmatrix} 0 \\ 4 \end{bmatrix}} \right\}, \quad O = \left\{ \underset{\vec{u}_1}{\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}}, \underset{\vec{u}_2}{\begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}} \right\}.$$

$$\underset{A}{\begin{bmatrix} 1 & 0 \\ 1 & 4 \end{bmatrix}} = \underset{Q}{\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}} \begin{bmatrix} \boldsymbol{w}_1 \cdot \boldsymbol{u}_1 & \boldsymbol{w}_2 \cdot \boldsymbol{u}_1 \\ 0 & \boldsymbol{w}_2 \cdot \boldsymbol{u}_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 2\sqrt{2} \\ 0 & 2\sqrt{2} \end{bmatrix} = Q\,R.$$

$\vec{w}_1 \cdot \vec{u}_1 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$

$\vec{w}_2 \cdot \vec{u}_1 = \begin{bmatrix} 0 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \frac{4}{\sqrt{2}} = 2\sqrt{2}$

$\vec{w}_2 \cdot \vec{u}_2 = \begin{bmatrix} 0 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \frac{4}{\sqrt{2}} = 2\sqrt{2}$

<u>Exercise 2</u>) Verify that R could have been recovered via the formula

$$Q^T A = R \, .$$

$$A = QR$$
$$Q^T A = Q^T Q R = IR = R$$

$$Q^T A = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} \sqrt{2} & 2\sqrt{2} \\ 0 & 2\sqrt{2} \end{bmatrix}$$

much easier

From previous page ...

$$\begin{bmatrix} 1 & 0 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 2\sqrt{2} \\ 0 & 2\sqrt{2} \end{bmatrix}.$$

Exercise 3) Verify that the $A = Q\,R$ factorization in this example may be further factored as

$$\begin{bmatrix} 1 & 0 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 2\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 0 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \quad , \quad , \quad , \quad , \quad , \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- So, the transformation $T(\underline{\boldsymbol{x}}) = A\,\underline{\boldsymbol{x}}$ is a composition of (1) an area-preserving shear, followed by (2) a diagonal scaling that increases area by a factor of $\sqrt{2} \cdot 2\sqrt{2} = 4$, followed by a rotation of $\frac{\pi}{4}$, which does not effect area. Since determinants of products matrices are the products of determinants (we checked this back when we studied determinants), and area expansion factors of compositions are also the products of the area expansion factors, the generalization of this example gives another explanation of why the determinant of A (or its absolute value in general) coincides with the area expansion factor, in the 2×2 case. You show in your homework that the only possible Q matrices in the 2×2 case are rotations as above, or reflections across lines through the origin. In the latter case, the determinant of Q is -1, and the determinant of A is negative.

Example from Exercise 1:.

$$B = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \right\} \qquad O = \left\{ \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Exercise 4a Find the $A = Q\,R$ factorization based on the data above, for

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 4 & -2 \\ 0 & 0 & 3 \end{bmatrix}, \quad Q = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A = QR$$

$$R = Q^T A = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 4 & -2 \\ 0 & 0 & 3 \end{bmatrix}$$

R.

$$\text{solution } A = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 2\sqrt{2} & -\frac{1}{\sqrt{2}} \\ 0 & 2\sqrt{2} & -\frac{3}{\sqrt{2}} \\ 0 & 0 & 3 \end{bmatrix}$$

Exercise 4b Further factor R into a diagonal matrix times a volume-preserving shear and interpret the transformation $T(\underline{\mathbf{x}}) = A\,\underline{\mathbf{x}}$ as a composition of (1) a volume preserving shear, followed by (2) a coordinate scaling that increases volume by a factor of 12, followed by a rotation about the x_3 axis in $\mathbb{R}^3$, which preserves volume. The generalization of this example gives another explanation of why the determinant of A (or its absolute value in general) is the volume expansion factor for the transformation $T(\underline{\mathbf{x}}) = A\,\underline{\mathbf{x}}$.

further factor